Exercises discussed on November 13, 2012

23. (Tower of Hanoi) Given a tower of n disks initially stacked in increasing order on one of three pegs, the task is to transfer the entire tower to one of the other pegs, moving only one disk at a time and never moving a larger disk onto a smaller one. Let a_n denote the minimal number of moves needed.

    Find a recurrence for a_n. Compute the first few values and guess a closed form solution. Derive the closed form solution using techniques from the lecture.

24. Given n people numbered from 1 to n sitting at a round table. Starting from person 1 in clockwise order every second person leaves until only one person remains (the first person to leave is person 2). Let $J(n)$ denote the number of the remaining person. Determine $J(n)$.

25. Prove Theorem 3.2 for recurrences of order two, i.e., for $c_0 \neq 0, c_1 \in \mathbb{K}$ with

    $$x^2 + c_1 x + c_0 = (x - \alpha_1)(x - \alpha_2)$$

    with $\alpha_1 \neq \alpha_2 \in \mathbb{K}$, show that $(\alpha_1^n)_{n\geq 0}, (\alpha_2^n)_{n\geq 0}$ form a basis for the solutions of the recurrence

    $$a_{n+2} + c_1 a_{n+1} + c_0 a_n = 0, \qquad n \geq 0,$$

    and that if $\alpha_1 = \alpha_2 = \alpha$ then $(\alpha^n)_{n\geq 0}, (n\alpha^n)_{n\geq 0}$ form a basis for the solutions of the above recurrence.

26. (Theorem 3.3) Show that a sequence $(a_n)_{n\geq 0}$ in $\mathbb{K}$ satisfies a C-finite recurrence

    $$a_{n+r} + c_{r-1} a_{n+r-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0, \qquad n \geq 0,$$

    with $c_i \in \mathbb{K}, c_0 \neq 0$, if and only if

    $$\sum_{n\geq 0} a_n x^n = \frac{p(x)}{1 + c_{r-1}x + \cdots + c_0 x^r}$$

    for some polynomial $p(x)$ with degree at most $r - 1$.