Exercises discussed on October 23, 2012

11. Show that the map $D_x : \mathbb{K}[[x]] \to \mathbb{K}[[x]]$ defined as

$$D_x \sum_{n=0}^{\infty} a_n x^n := \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$

turns $\mathbb{K}[[x]]$ into a differential ring.

12. Show that for the formal derivation and integration as defined in the lecture and for all $a(x) \in \mathbb{K}[[x]]$ it holds that

    (a) $D_x \int_x a(x) = a(x)$ (fundamental theorem of calculus I)

    (b) $\int_x D_x a(x) = a(x) - a(0)$ (fundamental theorem of calculus II)

    (c) $\langle x^n \rangle a(x) = \frac{1}{n!} \left(D_x^n a(x) \right) \big|_{x=0}$ (Taylor's formula)

13. Prove Theorem 2.10 (Multiplicative Inverse):
Let $a(x) \in \mathbb{K}[[x]]$. Then there exists a series $b(x) \in \mathbb{K}[[x]]$ with $a(x)b(x) = 1$ if and only if $a(0) \neq 0$.

14. Define the formal power series $\sum_{n \geq 0} b_n x^n = \left(\sum_{n=0} n! x^n \right)^{-1}$.

    (a) Verify that the coefficients b_7, b_8, b_9, b_{10} are negative.

    (b) Check what the *online encyclopedia of integer sequences* (Sloane's database) says about the sequence of coefficients $-b_1, -b_2, -b_3, -b_4, \ldots$