Exercises discussed on November 22, 2011

27. Prove Theorem 3.2 for recurrences of order two., i.e., for $c_0 \neq 0, c_1 \in \mathbb{K}$ with

$$x^2 + c_1 x + c_0 = (x - \alpha_1)(x - \alpha_2)$$

with $\alpha_1 \neq \alpha_2 \in \mathbb{K}$, show that $\left(\alpha_1^n\right)_{n\geq 0}, \left(\alpha_2^n\right)_{n\geq 0}$ form a basis of the solutions of the recurrence

$$a_{n+2} + c_1 a_{n+1} + c_0 a_n = 0,$$

and that if $\alpha_1 = \alpha_2$ then $\left(\alpha_1^n\right)_{n\geq 0}, \left(n\alpha_1^n\right)_{n\geq 0}$ form a basis of solutions of the above recurrence.

(BP2) Prove Theorem 3.2 for the general case.

28. (Theorem 3.3) Show thata sequence $\left(a_n\right)_{n\geq 0}$ satisfies a C-finite recurrence

$$a_{n+r} + c_{r-1} a_{n+r-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0, \quad n \geq 0,$$

with $c_i \in \mathbb{K}$, $c_0 \neq 0$ if and only if

$$\sum_{n\geq 0} a_n x^n = \frac{p(x)}{1 + c_{r-1}x + \cdots + c_1 x^{r-1} + c_0 x^r},$$

for some $p \in \mathbb{K}[x]$ with $\deg p(x) \leq r - 1$.

29. Use the GeneratingFunctions[1] package to

   - guess a recurrence for the sequence $\left(F_{2n}\right)_{n\geq 0}$ (you can use the Mathematica built-in function Fibonacci[n] to generate the data);
   - guess a recurrence for the Lucas numbers $\left(L_n\right)_{n\geq 0}$ (you can use the Mathematica built-in function LucasL[n] to generate the data);
   - compute the recurrences for $a_n = F_{2n} + L_n$ and $b_n = F_{2n} L_n$ using closure properties.

30. Show that the sequence of Harmonic numbers $\left(H_n\right)_{n\geq 0}$ is not C-finite.

[1] available at http://www.risc.jku.at/research/combinat/software/