Exercises discussed on November 8, 2011

17. Let $(a_n(x))_{n\geq 0}, (b_n(x))_{n\geq 0}$ be convergent sequences of formal power series with respective limits $a(x), b(x) \in \mathbb{K}[\![x]\!]$.

    Show that then also $(c_n(x))_{n\geq 0}$ with $c_n(x) = a_n(x) + b_n(x)$ is a convergent sequence of formal power series with limit $a(x) + b(x)$.

18. Prove the reflection formula, i.e., show that for x an indeterminate and $k \in \mathbb{N}$:

$$\binom{x}{k} = (-1)^k \binom{k-x-1}{k}.$$

19. Give a direct proof of the binomial theorem in the ring of formal power series, i.e., show that for $f(x) \in \mathbb{C}[\![x]\!]$ with $f(0) = 0$ and $\lambda \in \mathbb{C}$ that

$$(1 + f(x))^\lambda := \sum_{n\geq 0} \binom{\lambda}{n} f(x)^n \in \mathbb{C}[\![x]\!].$$

Recall the definition of *Stirling numbers of the second kind* $S_2(n, k)$ as the number of ways to partition an n-element set into a disjoint union of k nonempty subsets. They satisfy the recurrence relation

$$S_2(n, k) = S_2(n-1, k-1) + kS_2(n-1, k), \qquad n, k \geq 1,$$

with initial values $S_2(0, 0) = 0$ and $S_2(n, 0) = S_2(0, n) = 0$ for $n \geq 1$.

20. Show that for $k \in \mathbb{N}$

$$\sum_{n=0}^{\infty} S_2(n, k)x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

21. Show that for $k \in \mathbb{N}$

$$\sum_{n=0}^{\infty} S_2(n, k)\frac{x^n}{n!} = \frac{1}{k!}\left(\mathrm{e}^x - 1\right)^k,$$

    and that

$$\sum_{n,k=0}^{\infty} S_2(n, k)\frac{x^n}{n!}y^k = \exp\left(y(\mathrm{e}^x - 1)\right).$$