Rational general solutions of parametrizable first-order algebraic ODEs

Guest lecture in Computer Analysis

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Preliminaries

First-order algebraic ordinary differential equation

Throughout the slides:

$\mathcal{F}$ is an algebraically closed field of characteristic zero, x is an indeterminate, and we consider the derivation $' = {}^{\mathrm{d}}/_{\mathrm{d}x}$.

Definition

An *algebraic ordinary differential equation* (AODE) of order one is an ODE of the form

$$A(x, y, y') = 0,$$

where $A \in \mathcal{F}[x, u, v]$ such that $\deg_v(A) > 0$. We call A the *defining polynomial* of the AODE.

Henceforth, we assume that A is irreducible.

Differential algebra recap:
Denote by $\mathcal{F}(x)\{y\}$ the differential polynomial ring over the differential field $(\mathcal{F}(x),')$. Let $P \in \mathcal{F}(x)\{y\}$ be an irreducible differential polynomial of order one.

Ritt [Rit50] showed that the radical differential ideal generated by P can be decomposed into

$$\{P\} = (\{P\} : S_P) \cap \{P, S_P\},$$

where $S_P = {}^{\partial P}/_{\partial y'}$ denotes the separant of P. The component $\{P\} : S_P$ is an essential prime divisor of this decomposition, whereas $\{P, S_P\}$ may have to be further decomposed.

In the literature, the prime different ideal $\{P\} : S_P$ is called the *general component* of P. We consider the left-hand side of a first-order AODE as a differential polynomial.

Definition

A *general solution* of a first-order AODE $A(x, y, y') = 0$ is a generic zero of the general component of $A(x, y, y')$.

Definition

A general solution $\hat{y}$ of the first-order AODE $A(x, y, y') = 0$ is called a *rational general solution* if $\hat{y} \in \mathbb{F}(x)$, where $\mathbb{F} \supsetneq \mathcal{F}$ is a constant differential extension field.

The typical case in this presentation is $\mathbb{F} = \mathcal{F}(C)$, where C is an arbitrary constant.

Let $\mathcal{V} \subseteq \mathbb{A}^n(\mathcal{F})$ be a d-dimensional affine algebraic variety, i.e. an irreducible affine algebraic set.

Definition

A *proper rational parametrization* of $\mathcal{V}$ is a birational map

$$\begin{aligned} \Phi_{\mathcal{V}} : \mathbb{A}^d(\mathcal{F}) &\to \mathcal{V}, \\ \mathbf{t} &\mapsto (\Phi_1(\mathbf{t}), \dots, \Phi_n(\mathbf{t})), \end{aligned}$$

where $\Phi_i \in \mathcal{F}(\mathbf{t}) = \mathcal{F}(t_1, \dots, t_d)$ for all $1 \leq i \leq n$.

Varieties which possess a proper rational parametrization are called *rational* or a *rational variety*. In the sequel, proper rational parametrizations are denoted by a tuple of rational functions $\Phi_{\mathcal{V}} = (\Phi_1, \dots, \Phi_n)$.

Solution algorithms

Algorithms for parametrizable first-order AODEs

We shall (briefly) discuss the following algorithms for computing rational general solutions (RGSs) of first-order AODEs:

1. [NW10]: Computation of RGSs of first-order AODEs via surface-parametrization.
2. [VGW18]: Computation of (strong) RGSs of first-order AODEs via curve-parametrization.
3. [FG04]: Computation of RGSs of first-order autonomous AODEs.

Remark: All these methods derive a RGS from a suitable rational reparametrization of a proper rational parametrization of an associated affine algebraic variety.

Consider the first-order AODE $A(x, y, y') = 0$ with defining polynomial $A \in \mathcal{F}[x, u, v]$.[1] The zero-locus of A defines the surface

$$\mathcal{S}_A := \left\{(a_0, a_1, a_2) \in \mathbb{A}^3(\mathcal{F}) \mid A(a_0, a_1, a_2) = 0\right\}.$$

Definition

We call $\mathcal{S}_A$ the *associated surface* (of the AODE). If $\mathcal{S}_A$ has a proper rational parametrization $\boldsymbol{\varphi}_{\mathcal{S}_A} = (\varphi_0, \varphi_1, \varphi_2)$ with $\varphi_0, \varphi_1, \varphi_2 \in \mathcal{F}(t_1, t_2)$, then the AODE is called *surface-parametrizable*.

[1] Recall that A is irreducible and of positive degree in v.

Let $A(x, y, y') = 0$ be surface-parametrizable, $\boldsymbol{\varphi}_{\mathcal{S}_A} = (\varphi_0, \varphi_1, \varphi_2)$ a proper rational parametrization, and assume $\hat{y}$ is a rational (general) solution. Such a solution generates a parametric (family of) rational curve(s) $\mathcal{C}_{\hat{y}}(x) = (x, \hat{y}(x), \hat{y}'(x))$ on $\mathcal{S}_A$.

Let $(\sigma(x), \tau(x)) = \boldsymbol{\varphi}_{\mathcal{S}_A}^{-1}(x, \hat{y}(x), \hat{y}'(x))$. Application of $\boldsymbol{\varphi}_{\mathcal{S}_A}$ on both sides yields $\boldsymbol{\varphi}_{\mathcal{S}_A}(\sigma(x), \tau(x)) = (x, \hat{y}(x), \hat{y}'(x))$ which gives the following conditions for $\sigma(x)$ and $\tau(x)$:

$$\begin{cases} \varphi_0(\sigma(x), \tau(x)) = x \\ \varphi_2(\sigma(x), \tau(x)) = \dfrac{\mathrm{d}}{\mathrm{d}x} \varphi_1(\sigma(x), \tau(x)). \end{cases}$$

This system can be solved for $\sigma(x)$ and $\tau(x)$.

Definition

Let $\boldsymbol{\varphi}_{\mathcal{S}_A} = (\varphi_0, \varphi_1, \varphi_2)$, where $\varphi_0, \varphi_1, \varphi_2 \in \mathcal{F}(t_1, t_2)$, be a proper rational parametrization of the surface-parametrizable first-order AODE $A(x, y, y') = 0$.

The system

$$\begin{cases} \sigma' = \dfrac{\varphi_2(\sigma,\tau)\frac{\partial\varphi_0(\sigma,\tau)}{\partial\tau} - \frac{\partial\varphi_1(\sigma,\tau)}{\partial\tau}}{\frac{\partial\varphi_0(\sigma,\tau)}{\partial\tau}\frac{\partial\varphi_1(\sigma,\tau)}{\partial\sigma} - \frac{\partial\varphi_0(\sigma,\tau)}{\partial\sigma}\frac{\partial\varphi_1(\sigma,\tau)}{\partial\tau}} \\ \tau' = \dfrac{\varphi_2(\sigma,\tau)\frac{\partial\varphi_0(\sigma,\tau)}{\partial\sigma} - \frac{\partial\varphi_1(\sigma,\tau)}{\partial\sigma}}{\frac{\partial\varphi_0(\sigma,\tau)}{\partial\sigma}\frac{\partial\varphi_1(\sigma,\tau)}{\partial\tau} - \frac{\partial\varphi_0(\sigma,\tau)}{\partial\tau}\frac{\partial\varphi_1(\sigma,\tau)}{\partial\sigma}} \end{cases}$$

is called the *associated planar system* wrt. $\boldsymbol{\varphi}_{\mathcal{S}_A}$.

Question: What is a suitable rational reparametrization of $\boldsymbol{\varphi}_{\mathcal{S}_A}$?

Answer: A rational general solution $(\hat{\sigma}(x), \hat{\tau}(x))$ of the associated planar system wrt. $\boldsymbol{\varphi}_{\mathcal{S}_A}$.

In this case, $\hat{y} = \varphi_1(\hat{\sigma}(x+K), \hat{\tau}(x+K))$ is a rational general solution of the original AODE, where $K = x - \varphi_0(\hat{\sigma}(x), \hat{\tau}(x))$ is a constant. Thus, surface-parametrizable first-order AODEs can be transformed into autonomous planar rational systems. This transformation preserves rational general solutions.

Theorem ([NW11a])

The rational general solutions of a surface-parametrizable first-order AODE are in one-to-one correspondence with the rational general solutions of its associated planar system.

Algorithm 1: RGS surface-param. first-order AODE [NW10]

Input : First-order AODE $A(x, y, y') = 0$

Output: Rational general solution $\hat{y}$ or string message

if *the associated surface* $\mathcal{S}_A$ *is rational* **then**

  Compute a proper rational parametrization

$$\varphi_{\mathcal{S}_A} = (\varphi_0, \varphi_1, \varphi_2), \text{ where } \varphi_0, \varphi_1, \varphi_2 \in \mathcal{F}(t_1, t_2).$$

  Find a rational general solution $(\hat{\sigma}(x), \hat{\tau}(x))$ of

$$\begin{cases} \sigma' = \dfrac{\varphi_2(\sigma,\tau)\frac{\partial \varphi_0(\sigma,\tau)}{\partial \tau} - \frac{\partial \varphi_1(\sigma,\tau)}{\partial \tau}}{\frac{\partial \varphi_0(\sigma,\tau)}{\partial \tau}\frac{\partial \varphi_1(\sigma,\tau)}{\partial \sigma} - \frac{\partial \varphi_0(\sigma,\tau)}{\partial \sigma}\frac{\partial \varphi_1(\sigma,\tau)}{\partial \tau}} \\ \tau' = \dfrac{\varphi_2(\sigma,\tau)\frac{\partial \varphi_0(\sigma,\tau)}{\partial \sigma} - \frac{\partial \varphi_1(\sigma,\tau)}{\partial \sigma}}{\frac{\partial \varphi_0(\sigma,\tau)}{\partial \sigma}\frac{\partial \varphi_1(\sigma,\tau)}{\partial \tau} - \frac{\partial \varphi_0(\sigma,\tau)}{\partial \tau}\frac{\partial \varphi_1(\sigma,\tau)}{\partial \sigma}}. \end{cases}$$

  if *no such solution exists* **then**

    return *"AODE has no rational general solution"*

  else

    return $\hat{y} = \varphi_1(\hat{\sigma}(x+K), \hat{\tau}(x+K))$, *where* $K = x - \varphi_0(\hat{\sigma}(x), \hat{\tau}(x))$

else

  return *"AODE is not surface-parametrizable"*

Example

Consider the first-order AODE $x^3y' - y^2 - x^2y = 0$ and let $\mathcal{F} = \overline{\mathbb{Q}}$. The associated surface

$$\mathcal{S}_A = \left\{(a_0, a_1, a_2) \in \mathbb{A}^3\left(\overline{\mathbb{Q}}\right) \mid a_0^3\, a_2 - a_1^2 - a_0^2\, a_1 = 0\right\}$$

can be parametrized by

$$\boldsymbol{\varphi}_{\mathcal{S}_A} = \left(\varphi_0 = t_1, \varphi_1 = t_2, \varphi_2 = \frac{t_2^2 + t_2 t_1^2}{t_1^3}\right).$$

The associated planar system wrt. $\boldsymbol{\varphi}_{\mathcal{S}_A}$ is

$$\begin{cases} \sigma' = 1 \\ \tau' = \frac{\tau^2 + \tau\sigma^2}{\sigma^3}, \end{cases}$$

with the solution $(\hat{\sigma}(x) = x, \hat{\tau}(x) = x^2/(Cx+1))$. This yields the rational general solution $\hat{y} = \varphi_1(\hat{\sigma}(x+K), \hat{\tau}(x+K)) = x^2/(Cx+1)$, where $K = x - \varphi_0(\hat{\sigma}(x), \hat{\tau}(x)) = 0$.

Curve-parametrizable first-order AODEs

Consider a first-order AODE $A(x, y, y') = 0$ and view its defining polynomial $A \in \mathcal{F}(x)[u, v]$. The zero-locus of A defines the curve

$$\mathcal{C}_A := \left\{(a_1, a_2) \in \mathbb{A}^2\left(\overline{\mathcal{F}(x)}\right) \mid A(a_1, a_2) = 0\right\}.$$

Note: The polynomial A remains irreducible in $\mathcal{F}(x)[u, v]$, but might factor over $\overline{\mathcal{F}(x)}$.

Definition

We call $\mathcal{C}_A$ the *associated curve* (of the AODE). If $\mathcal{C}_A$ has a proper rational parametrization $\boldsymbol{\psi}_{\mathcal{C}_A} = (\psi_1, \psi_2)$ such that $\psi_1, \psi_2 \in \mathcal{F}(x)(t)$, then the AODE is called *curve-parametrizable*.

Note: The coefficients of the parametrization in the previous definition must be chosen from $\mathcal{F}(x)$ (not from $\overline{\mathcal{F}(x)}$).

Let $\boldsymbol{\psi}_{\mathcal{C}_A} = (\psi_1, \psi_2)$ with $\psi_1, \psi_2 \in \mathcal{F}(x)(t)$ be a proper rational parametrization of the curve-parametrizable AODE $A(x, y, y') = 0$ and let $\hat{y}$ be a rational (general) solution. In this case, $(\hat{y}, \hat{y}')$ defines a (family of) point(s) on the associated curve $\mathcal{C}_A$.

Let $\omega = \boldsymbol{\psi}_{\mathcal{C}_A}^{-1}(\hat{y}, \hat{y}')$, then $\boldsymbol{\psi}_{\mathcal{C}_A}(\omega) = (\hat{y}, \hat{y}')$ and we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}x}\psi_1(\omega) = \psi_2(\omega).$$

A solution for ω solves a quasi-linear equation.

Definition

Let $\boldsymbol{\psi}_{\mathcal{C}_A} = (\psi_1, \psi_2)$, where $\psi_1, \psi_2 \in \mathcal{F}(x)(t)$, be a proper rational parametrization of the curve-parametrizable first-order AODE $A(x, y, y') = 0$.

We call the quasi-linear ODE

$$\omega' = \frac{\psi_2(\omega) - \frac{\partial \psi_1(\omega)}{\partial x}}{\frac{\partial \psi_1(\omega)}{\partial \omega}}$$

the *associated quasi-linear equation* wrt. $\boldsymbol{\psi}_{\mathcal{C}_A}$.

Question: What is a suitable rational reparametrization of $\boldsymbol{\psi}_{\mathcal{C}_A}$?

Answer: A rational general solution $\hat{\omega}$ of the associated quasi-linear equation.

In this case, $\hat{y} = \psi_1(\hat{\omega})$ is a rational general solution of the original AODE. Any curve-parametrizable first-order AODE can be transformed into a quasi-linear ODE and this transformation preserves rational general solutions.

Theorem ([VGW18])

There is a one-to-one correspondence between the rational general solutions of a curve-parametrizable first-order AODE and the rational general solutions of its associated quasi-linear equation.

Algorithm 2: RGS curve-param. first-order AODE [VGW18] (simplified version)

Input : First-order AODE $A(x, y, y') = 0$

Output: Rational general solution $\hat{y}$ or string message

if *the associated curve* $\mathcal{C}_A$ *is rational* **then**

  Compute a proper rational parametrization

$$\psi_{\mathcal{C}_A} = (\psi_1, \psi_2) \text{ such that } \psi_1, \psi_2 \in \mathcal{F}(x)(t).$$

  Find a rational general solution $\hat{\omega}$ of

$$\omega' = \frac{\psi_2(\omega) - \frac{\partial \psi_1(\omega)}{\partial x}}{\frac{\partial \psi_1(\omega)}{\partial \omega}}.$$

  if *no such solution exists* **then**

    return *"AODE has no rational general solution"*

  else

    return $\hat{y} = \psi_1(\hat{\omega})$

else

  return *"AODE is not curve-parametrizable"*

Example

Once again, consider the first-order AODE $x^3y' - y^2 - x^2y = 0$ with $\mathcal{F} = \overline{\mathbb{Q}}$. The associated curve

$$\mathcal{C}_A := \left\{(a_1, a_2) \in \mathbb{A}^2\left(\overline{\mathbb{Q}(x)}\right) \mid x^3\, a_2 - a_1^2 - x^2\, a_1 = 0\right\}.$$

is irreducible and can be parametrized by

$$\boldsymbol{\psi}_{\mathcal{C}_A} = (\psi_1 = x^2(xt-1), \psi_2 = x^2t(xt-1)).$$

The associated quasi-linear equation

$$\omega' = \frac{2}{x^2} - \frac{4}{x}\omega + \omega^2$$

has the rational general solution $\hat{\omega} = {}^{2}/_{x} + {}^{1}/_{(C-x)}$. From this we obtain the solution $\hat{y} = {}^{(Cx^2)}/_{(C-x)}$.

Autonomous first-order AODEs

The case of an autonomous first-order AODE $A(y, y') = 0$ is particularly easy. In this setting the zero-locus of A defines the curve

$$\mathcal{C}_A^a := \{(a_1, a_2) \in \mathbb{A}^2(\mathcal{F}) \mid A(a_1, a_2) = 0\}.$$

Definition

We call $\mathcal{C}_A^a$ the *associated autonomous curve* (of the AODE).

Let $\chi_{\mathcal{C}_A^a} = (\chi_1, \chi_2)$, where $\chi_1, \chi_2 \in \mathcal{F}(t)$, be a proper rational parametrization of the associated autonomous curve of an autonomous first-order AODE $A(y, y') = 0$.

If this parametrization is if the form:

1. $\chi_2 / \frac{\partial \chi_1}{\partial t} = a \in \mathcal{F}$: Then $a(x + C)$ is a suitable rational reparametrization and $\hat{y} = \chi_1(a(x + C))$ is a rational general solution of the original AODE.
2. $\chi_2 / \frac{\partial \chi_1}{\partial t} = a(t - b)^2 \in \mathcal{F}[t]$, $a \neq 0$: Then $b - 1/(a(x + C))$ is a suitable rational reparametrization and $\hat{y} = \psi_1(b - 1/(a(x + C)))$ is a rational general solution of the original AODE.

Otherwise, $A(y, y') = 0$ has no rational general solution [FG04].[2]

[2]Except for the trivial differential equation $cy' = 0$, where $c \in \mathcal{F} \setminus \{0\}$.

Algorithm 3: RGS autonomous first-order AODE [FG04] (simplified version)

Input : Autonomous first-order AODE $A(y, y') = 0$

Output: Rational general solution $\hat{y}$ or string message

if *the associated autonomous curve* $\mathcal{C}_A^a$ *is rational* **then**

  Compute a proper rational parametrization

$$\chi_{\mathcal{C}_A^a} = (\chi_1, \chi_2), \text{ where } \chi_1, \chi_2 \in \mathcal{F}(t)$$

  and let $\mu = \chi_2 / \frac{\partial \chi_1}{\partial t}$.

  if $\mu = a \in \mathcal{F} \setminus \{0\}$ **then**

    return $\hat{y} = \chi_1(a(x + C))$

  else if $\mu = a(t - b)^2$ *for some* $a, b \in \mathcal{F}$, $a \neq 0$ **then**

    return $\hat{y} = \chi_1\Big(b - \frac{1}{a(x+C)}\Big)$

  else

    return *"AODE has no rational general solution"*

else

  return *"AODE has no rational general solution"*

Example

Consider the autonomous first-order AODE $y'^2 - 4y^3 = 0$ with $\mathcal{F} = \overline{\mathbb{Q}}$. The associated autonomous curve

$$\mathcal{C}_A^a := \left\{(a_1, a_2) \in \mathbb{A}^2\left(\overline{\mathbb{Q}}\right) \mid a_2^2 - 4a_1^3 = 0\right\}.$$

has the proper rational parametrization

$$\chi_{\mathcal{C}_A^a} = \left(\chi_1 = \frac{t^2}{4}, \chi_2 = \frac{t^3}{4}\right).$$

We see that $\mu = \chi_2 / \frac{\partial \chi_1}{\partial t} = t^2/2$ is of the second form with $a = 1/2$ and $b = 0$. From this we obtain the rational general solution $\hat{y} = \chi_1(b - 1/(a(x+C))) = 1/(x+C)^2$.

Summary

Three methods for finding rational general solutions of first-order AODEs were discussed:

- **Method 1** derives a solution from a proper rational parametrization of an associated surface by solving an associated planar rational system.
- **Method 2** computes a proper rational parametrization of a curve over a rational function field and derives a solution by solving an associated quasi-linear equation.
- **Method 3** is a special case for autonomous first-order AODEs with a particularly easy reparametrization condition.

These methods are implemented in the Maple package `AGADE` which is available at:
`https://github.com/JohannMitteramskogler/AGADE`.

Demo

Comparison of the methods

Let the field $\mathcal{F}$ be fixed.

Definition

Denote by $\mathbf{A}_{ODE}$ the class of all first-order AODEs. Furthermore, let $\mathbf{A}_{ODE}^{(SP)}$ and $\mathbf{A}_{ODE}^{(CP)}$ denote the subclasses of first-order AODEs which are surface- and curve-parametrizable, respectively.

How are these classes related?

Remark ([MW22])

Every curve-parametrizable first-order AODE is surface-parametrizable and the inclusion is strict. In other words $\mathbf{A}_{ODE}^{(CP)} \subsetneq \mathbf{A}_{ODE}^{(SP)}$.

Definition

Denote by $\mathbf{A}_{ODE}^{(RGS)}$ the subclass of $\mathbf{A}_{ODE}$ that possess a rational general solution.

A rational general solution contained in $\mathcal{F}(C)(x)$, where C is an arbitrary constant, is called a *strong rational general solution*.

Definition

Denote by $\mathbf{A}_{ODE}^{(SRGS)}$ the subclass of $\mathbf{A}_{ODE}^{(RGS)}$ that possess a strong rational general solution.

Theorem ([VGW18])

The class of first-order AODEs with a strong rational general solution coincides with the class of curve-parametrizable first-order AODEs that possess a rational general solution. In terms of the introduced notation

$\mathbf{A}_{ODE}^{(SRGS)} = \mathbf{A}_{ODE}^{(CP)} \cap \mathbf{A}_{ODE}^{(RGS)}$.

What about surface-parametrizable first-order AODEs? Since the class of surface-parametrizable AODEs is strictly larger than the class of curve-parametrizable AODEs, can we find more solutions with this approach?

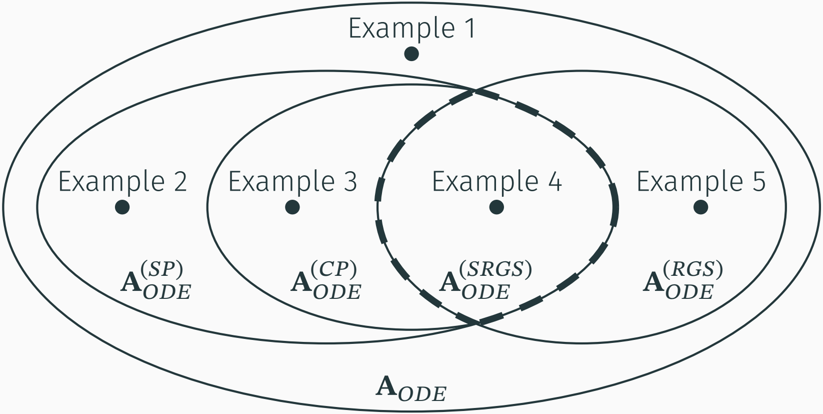
Theorem ([MW22])

If a surface-parametrizable first-order AODE has a rational general solution then the AODE has a strong rational general solution.

Corollary

The class of first-order AODEs with a strong rational general solution coincides with the class of surface-parametrizable first-order AODEs that possess a rational general solution, i.e. $\mathbf{A}_{ODE}^{(SRGS)} = \mathbf{A}_{ODE}^{(SP)} \cap \mathbf{A}_{ODE}^{(RGS)}$.

The full picture



Example first-order AODEs

Example 1: $y'^2 + y^3 + 1 = 0$

Example 2: $y'^2 - y^3 - x = 0$

Example 3: $y' - y = 0$

Example 4: $y' - y^2 = 0$ and $\hat{y} = 1/(C - x)$ is a (strong) rational general solution

Example 5: $x^2\, y'^2 - 2\, x\, y\, y' - y'^3 + y^2 - 2 = 0$ and $\hat{y} = C\, x + \sqrt{C^3 + 2}$ is a rational general solution

Final remark

One cannot hope to find a rational general solution of a surface-parametrizable first-order AODE if the associated curve $\mathcal{C}_A$ is not rational. Given a proper rational parametrization $\boldsymbol{\psi}_{\mathcal{C}_A} = (\psi_1, \psi_2)$ such that $\psi_1, \psi_2 \in \mathcal{F}(x)(t)$, let $\boldsymbol{\varphi}_{\mathcal{S}_A} = (\varphi_0 = x, \varphi_1 = \psi_1, \varphi_2 = \psi_2)$ be the corresponding surface parametrization. Due to the special shape of $\boldsymbol{\varphi}_{\mathcal{S}_A}$ the associated planar system simplifies to

$$\begin{cases} \sigma' = \dfrac{-\frac{\partial \varphi_1(\sigma,\tau)}{\partial \tau}}{-\frac{\partial \varphi_1(\sigma,\tau)}{\partial \tau}} = 1 \\[2ex] \tau' = \dfrac{\varphi_2(\sigma,\tau) - \frac{\partial \varphi_1(\sigma,\tau)}{\partial \sigma}}{\frac{\partial \varphi_1(\sigma,\tau)}{\partial \tau}}. \end{cases}$$

This system actually is equivalent to the associated quasi-linear equation wrt. $\boldsymbol{\psi}_{\mathcal{C}_A}$.

Thank you!

References

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Appendix

Solutions of the associated planar system

Given an autonomous planar rational system

$$\begin{cases} \sigma' = \dfrac{M_\sigma}{N_\sigma} \\ \tau' = \dfrac{M_\tau}{N_\tau}, \end{cases}$$

where $M_\sigma, M_\tau \in \mathcal{F}[\sigma, \tau]$ and $N_\sigma, N_\tau \in \mathcal{F}[\sigma, \tau] \setminus \{0\}$. We may assume w.l.o.g. that numerators and denominators are coprime.

Definition

A *rational first integral* of the autonomous planar rational system is a rational function $F \in \mathcal{F}(\sigma, \tau) \setminus \mathcal{F}$ such that

$$\frac{M_\sigma}{N_\sigma}\frac{\partial F}{\partial \sigma} + \frac{M_\tau}{N_\tau}\frac{\partial F}{\partial \tau} = 0.$$

Solutions of such systems can be obtained by finding (rational) first integrals. A rational first integral $F = {}^{P}\!/_{Q}$, where P and Q are coprime, gives rise to an invariant algebraic curve.

If an irreducible factor of such an invariant algebraic curve can be (properly) parametrized, then we have an algorithm for finding rational general solutions.

Algorithm 4: RGS planar rational system [NW11b]

Input : Autonomous planar rational system

Output: Rational general solution $(\hat{\sigma}, \hat{\tau})$ or string message

Compute a rational first integral $F = {}^{P}/_{Q} \in \mathcal{F}(\sigma, \tau) \setminus \mathcal{F}$ of the input system, where $P, Q \in \mathcal{F}[\sigma, \tau]$ are coprime.

if *no such rational first integral exists* **then**

  return *"Planar rational system has no rational general solution"*

else

  Let C be a transcendental constant. Take any irreducible factor I of $P - KQ \in \overline{\mathcal{F}(C)}[\sigma, \tau]$ and let $\mathcal{C}_I$ be the plane curve defined by I.

  if $\mathcal{C}_I$ *is rational* **then**

    Compute a proper rational parametrization

$$\psi_{\mathcal{C}_I} = (\psi_1(t), \psi_2(t)), \text{ where } \psi_1, \psi_2 \in \overline{\mathcal{F}(C)}(t).$$

    Find a linear rational function $\hat{\omega} \in \overline{\mathcal{F}(C)}(x)$ that solves either

$$\omega' = \frac{1}{{}^{\partial\psi_1(\omega)}/_{\partial\omega}} \frac{M_\sigma(\psi_1(\omega), \psi_2(\omega))}{N_\sigma(\psi_1(\omega), \psi_2(\omega))} \text{ or } \omega' = \frac{1}{{}^{\partial\psi_2(\omega)}/_{\partial\omega}} \frac{M_\tau(\psi_1(\omega), \psi_2(\omega))}{N_\tau(\psi_1(\omega), \psi_2(\omega))}.$$

    if *such a linear rational function exists* **then**

      return $(\hat{\sigma}, \hat{\tau}) = (\psi_1(\hat{\omega}), \psi_2(\hat{\omega}))$

  return *"Planar rational system has no rational general solution"*