Formal Power Series Solutions of Algebraic Ordinary Differential Equations

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The content of this talk has mainly taken from the papers

- J. Cano and S. Falkensteiner and J.R. Sendra, *Existence and Convergence of Puiseux Series Solutions for First Order Autonomous Differential Equations*. Journal of Symbolic Computation (in print), 2020.
- S. Falkensteiner, N. Thieu Vo, Y. Zhang, *On Formal Power Series Solutions of Algebraic Ordinary Differential Equations*. 2020. arxiv.org/abs/1803.09646

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Introduction

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$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{1}$$

where $F \in \mathbb{K}[x, y, \ldots, y^{(n)}]$ is of order n.

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$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{1}$$

where $F \in \mathbb{K}[x, y, \ldots, y^{(n)}]$ is of order n. Of particular interest will be the case where $n = 1$ and F has constant coefficients, i.e. $F \in \mathbb{K}[y, y']$.

Motivation and background

Goal

Given a single AODE $F(x, y, \ldots, y^{(n)}) = 0$, find all formal power series solutions, i.e. all $\tilde{y}(x) = \sum_{i \geq 0} \frac{c_i}{i!} x^i \in \mathbb{K}[[x]]$ such that

$$F(x, \tilde{y}(x), \ldots, \tilde{y}^{(n)}(x)) = 0.$$

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Example 1

Consider the AODE

$$F = y'^2 - y^3 - y^2 = 0,$$

which has no rational general solution, but infinitely many formal power series solutions such as $\tilde{y}(x) = \tanh\left(\frac{c-x}{2}\right)^2 - 1$.

Motivation and background

E.L. Ince (1926): gives full understanding of linear ODEs.
J. Denef and L. Lipshitz (1984): find generic solutions of AODEs of any order.
Newton-Puiseux technique for AODEs: finds formal Puiseux series solutions of AODEs of any order, but is not completely algorithmic.

Direct approach

We use the notation $[x^k]\tilde{y}(x)$ to refer to the coefficient of x^k of a formal power series $\tilde{y}(x)$.

Lemma 1

Let $\tilde{y} = \sum_{i \geq 0} \frac{c_i}{i!} x^i \in \mathbb{K}[[x]]$. Then $\tilde{y}(x)$ is a FPSS of $F(x, y, \ldots, y^{(n)}) = 0$ iff

- $[x^0]F(x, \tilde{y}(x), \ldots, \tilde{y}^{(n)}(x)) = F(0, c_0, \ldots, c_n) = 0$.
- $[x^k]F(x, \tilde{y}(x), \ldots, \tilde{y}^{(n)}(x)) = [x^0]F^{(k)}(x, \tilde{y}(x), \ldots, \tilde{y}^{(k)}(x)) = F^{(k)}(0, c_0, \ldots, c_{n+k}) = 0$ for every $k \geq 1$.

Direct approach

Ritt's Lemma

Let $F \in \mathbb{K}[x, y, \dots, y^{(n)}]$. For every $k \geq 1$ there exists $R_k \in \mathbb{K}[x, y, \dots, y^{(n+k-1)}]$ such that

$$F^{(k)} = \frac{\partial F}{\partial y^{(n)}} \cdot y^{(n+k)} + R_k.$$

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$$F^{(k)} = \frac{\partial F}{\partial y^{(n)}} \cdot y^{(n+k)} + R_k.$$

Let $\tilde{y}(x) = \sum_{i \geq 0} \frac{c_i}{i!} x^i$ be a FPSS and $\frac{\partial F}{\partial y^{(n)}}(0, c_0, \ldots, c_n) \neq 0$.
Then, for $k \geq 1$,

$$\begin{aligned}
0 &= [x^0] F^{(k)}(x, \tilde{y}(x), \ldots, \tilde{y}^{(n+k)}(x)) \\
&= [x^0] \frac{\partial F(x, \tilde{y}(x), \ldots, \tilde{y}^{(n)}(x))}{\partial y^{(n)}} \cdot \tilde{y}^{(n+k)}(x) + [x^0] R_k(x, \tilde{y}(x), \ldots, \tilde{y}^{(n+k-1)}(x)) \\
&= \frac{\partial F(0, c_0, \ldots, c_n)}{\partial y^{(n)}} \cdot c_{n+k} + R_k(0, c_0, \ldots, c_{k+n-1})
\end{aligned}$$

Direct approach

Equivalently,

$$c_{n+k} = \frac{-R_k(0, c_0, \ldots, c_{n+k-1})}{\frac{\partial F(0,c_0,\ldots,c_n)}{\partial y^{(n)}}}.$$

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Regular formal power series solutions

To conclude, for a given AODE $F \in \mathbb{K}[x, y, \ldots, y^{(n)}]$, the formal power series solutions $\tilde{y}(x) = \sum_{i \geq 0} \frac{c_i}{i!} x^i$ with $\frac{\partial F}{\partial y^{(n)}}(0, c_0, \ldots, c_n) \neq 0$ can be computed iteratively.

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Question: Can we say something about the solutions where $\frac{\partial F}{\partial y^{(n)}}(0, c_0, \ldots, c_n) = 0$?

Example 2

Consider the following AODE of order two:

$$F = x\, y'' - 3y' + x^2 y^2 = 0.$$

The separant is

$$\frac{\partial F}{\partial y^{(n)}}(0, c_0, c_1, c_2) = 0$$

for every initial values c_0, c_1, c_2.

Direct approach

Ritt's formula can be refined as follows. For a differential polynomial $F \in \mathbb{K}[x, y, \ldots, y^{(n)}]$ and $k, m \in \mathbb{N}$, we define

$$f_i = \begin{cases} \frac{\partial F}{\partial y^{(i)}}, & i = 0, \ldots, n; \\ 0, & \text{otherwise}; \end{cases}, \quad \mathcal{B}_m(k) = \left[\binom{k}{0} \quad \binom{k}{1} \quad \ldots \quad \binom{k}{m} \right],$$

$$\mathcal{S}_{F,m} = \begin{bmatrix} f_n & f_{n-1} & f_{n-2} & \cdots & f_{n-m} \\ 0 & f_n^{(1)} & f_{n-1}^{(1)} & \cdots & f_{n-m+1}^{(1)} \\ 0 & 0 & f_n^{(2)} & \cdots & f_{n-m+2}^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & f_n^{(m)} \end{bmatrix}, \quad Y_m = \begin{bmatrix} y^{(m)} \\ y^{(m-1)} \\ \vdots \\ y \end{bmatrix}.$$

The $\mathcal{S}_{F,m}$ are called *m-th separant matrix* of F.

Direct Approach

Refinement of Ritt's Lemma

Let $F \in \mathbb{K}[x, y, \ldots, y^{(n)}]$. Then for each $m \in \mathbb{N}$ and $k > 2m$ there exists a differential polynomial $r_{n+k-m-1}$ with order less than or equal to $n + k - m - 1$ such that

$$F^{(k)} = \mathcal{B}_m(k) \cdot \mathcal{S}_{F,m} \cdot Y_m^{(n+k-m)} + r_{n+k-m-1}. \qquad (2)$$

Direct Approach

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Almost all formal power series solutions

The formal power series solutions $\tilde{y}(x) = \sum_{i \geq 0} \frac{c_i}{i!} x^i$ with $\mathcal{S}_{F,m}(0, c_0, \dots, c_{n+m}) \neq \mathbf{0}$ can be computed iteratively.

Example 2

For $F = x\,y'' - 3y' + x^2y^2 = 0$ a zero is $\mathbf{c} = (0, c_0, 0, c_2)$ where c_0, c_2 are arbitrary constants in $\mathbb{K}$. The first separant matrix is

$$\mathcal{S}_{F,1}(\mathbf{c}) = \left[\frac{\partial F}{\partial y^{(2)}}(\mathbf{c})\right] = \left[0\right].$$

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$$\mathcal{S}_{F,1}(\mathbf{c}) = \left[\frac{\partial F}{\partial y^{(2)}}(\mathbf{c})\right] = \left[0\right].$$

For $F^{(1)} = x\,y''' - 2y'' + 2x^2y'y + 2xy^2$ and $F^{(2)} = 0$, we uniquely extend the initial value to $\mathbf{c} = (0, c_0, 0, 0, 2c_0^2)$. Then, the second separant matrix is

$$\mathcal{S}_{F,2}(\mathbf{c}) = \begin{bmatrix} \frac{\partial F}{\partial y^{(2)}}(\mathbf{c}) & \frac{\partial F}{\partial y^{(1)}}(\mathbf{c}) \\ 0 & \left(\frac{\partial F}{\partial y^{(2)}}\right)^{(1)}(\mathbf{c}) \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 0 & 1 \end{bmatrix}.$$

Using the refinement of Ritt's formula, we obtain that every $F^{(k)}$ with $k > 2$ can be written as

$$F^{(k)}(\mathbf{c}) = -3c_{k+1} + kc_{k+1} + r_k(0, c_0, \dots, c_k)$$

and hence,

$$c_{k+1} = \frac{r_k(0, c_0, \dots, c_k)}{k-3}.$$

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The numerator could be zero iff $k = 3$, which we exclude. Checking $F^{(3)}(\mathbf{c}) \equiv 0$, which is the case for arbitrary $c_4 \in \mathbb{K}$, we find the family of solutions

$$y(x) \equiv c_0 + \frac{c_0^2}{3}\,x^3 + \frac{c_4}{24}\,x^4 - \frac{c_0^3}{18}\,x^6 - \frac{c_0\,c_4}{252}\,x^7 - \frac{c_0^2\,c_4}{3024}\,x^{10} \mod x^{11}.$$

Direct Approach

Although for every differential polynomial $F \in \mathbb{K}[x, y, \ldots, y^{(n)}]$ and FPSS $\tilde{y}(x) = \sum_{i \geq 0} \frac{c_i}{i!} x^i$ (or a reduced equation of F) there is $m \in \mathbb{N}$ such that $\mathcal{S}_{F,m}(0, c_0, c_1, \ldots) \neq \mathbf{0}$, this process is not completely algorithmic. In other words, the m can in general not be bounded a-priori.

Algebro-geometric approach

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By considering y and y' as independent variables (y and z), F defines a plane affine algebraic curve

$$\mathcal{C}(F) = \{(a, b) \in \mathbb{K}^2 \mid F(a, b) = 0\}.$$

$\mathcal{C}(F)$ is called the corresponding curve of F.

Local parametrizations

A pair $\mathcal{P} \in K((t))^2$ is called a local parametrization of $\mathcal{C}(F)$ if

$$F(\mathcal{P}) = 0$$

holds and at least one component is non-constant. $\mathcal{P}(0)$ is called the center of $\mathcal{P}$.

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Local parametrizations can be computed by the Newton-Puiseux algorithm and are well-understood.

Newton-Puiseux algorithm

The field $\mathbb{K}\langle\langle t\rangle\rangle = \bigcup_{n\in\mathbb{N}^*} \mathbb{K}((t^{1/n}))$ is called field of formal Puiseux series and we call for $\varphi(t) \in \mathbb{K}\langle\langle t\rangle\rangle$ the minimal $n \in \mathbb{N}$ such that $\varphi(t) \in \mathbb{K}((t^{1/n}))$ the ramification index.

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Starting with an algebraic equation $F(y, z) = 0$, we can compute its solutions in z as formal Puiseux series (w.l.o.g. expanded around 0):

Strategy

- Consider for $F(y, z) = \sum_{i,j\geq 0} f_{i,j} y^i z^j$ the left part of the convex hull of $\{(i, j) \mid f_{i,j} \neq 0\}$ (the Newton polygon of F.)
- Take a side with slope $-1/\mu$ with points $(i_1, j_1), \ldots, (i_n, j_n)$ and $n \geq 2$ lying on it.
- Compute c from
$$\sum_{k=1}^{n} f_{i_k,j_k} c^{j_k} = 0.$$
- Repeat the process with $z \mapsto z + cx^\mu$.

Example 1

Let us consider again $F = z^2 - y^3 - y^2$. By looking at the Newton polygon of F we obtain $\mu = 1$ and $c^2 - 1 = 0$, hence, $c = \pm 1$.

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Newton-Puiseux algorithm

In each step we take a slope $\mu_k > \mu_{k-1}$ with $\mu_k \in \mathbb{Q}$. There is a bound for the number of steps from which on μ_k and c_k are unique and the denominator of μ_k does not increase anymore.

Newton-Puiseux algorithm

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Given a Puiseux-expansion $(y, P((y - c_0)^{1/n}))$, a local parametrization gets computed easily as

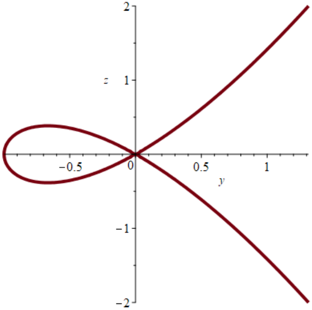
$$((t + c_0)^n, P(t)) \in \mathbb{K}((t))^2.$$

Places

Let **LocalPar**$(\mathcal{C}(F))$ denote the set of all local parametrizations of $\mathcal{C}(F)$. For $\mathcal{P}_1, \mathcal{P}_2 \in$ **LocalPar**$(\mathcal{C}(F))$ we define the equivalence relation $\mathcal{P}_1 \sim \mathcal{P}_2$ iff there exists $S \in \mathbb{K}[[t]]$ with $\mathrm{ord}(S) = 1$ such that $\mathcal{P}_1(S) = \mathcal{P}_2$.
We work with elements in **LocalPar**$(\mathcal{C}(F))/\sim$.

Example 1

For $F = z^2 - y^3 - y^2$ the curve looks as follows.



Local parametrizations at the origin can be given by

$$(t, \pm t - t^2/2 + \mathcal{O}(t^3)).$$

Places

Let $\mathcal{P} \in \mathbf{LocalPar}(\mathcal{C}(F))$. If there exists another $\mathcal{P}^* \in \mathbf{LocalPar}(\mathcal{C}(F))$ and $r > 1$ with $\mathcal{P} = \mathcal{P}^*(t^r)$ we say that $\mathcal{P}$ is reducible.

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A place is an equivalence class in $\textbf{LocalPar}(\mathcal{C}(F))/\sim$ of an irreducible local parametrization. The common center point is the center of the place.

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A place is an equivalence class in $\textbf{LocalPar}(\mathcal{C}(F))/\sim$ of an irreducible local parametrization. The common center point is the center of the place.

A place is the algebraic version of a branch:
If $\mathbb{K} = \mathbb{C}$, there exists for every branch a representative which components are analytic (in a certain neighborhood of 0).

First result

Lemma [Necessary Condition]

Let $\tilde{y}(x) \in \mathbb{K}\langle\langle x \rangle\rangle$ be a non-constant formal Puiseux series solution of $F(y, y') = 0$ with ramification index equals n. Then

$$(a(t), b(t)) = (\tilde{y}(t^n), \tilde{y}'(t^n)) \in \mathbf{LocalPar}(\mathcal{C}(F))$$

is an irreducible place centered at $(\tilde{y}(0), \tilde{y}'(0))$.

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$$n = \mathrm{ord}_t(a(t) - y_0) - \mathrm{ord}_t(b(t)).$$

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Strategy

Find the centers and places of $\mathcal{C}(F)$ containing a solution parametrization.

Order-suitability

Let $(a, b) \in \mathbf{LocalPar}(\mathcal{C}(F))$. We say that (a, b) is order-suitable if

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Note that order-suitability is independent of the representative of the place.

Theorem [Necessary and Sufficient Condition]

Let $\mathcal{P}$ be a place of $\mathcal{C}(F)$. Then $\mathcal{P}$ is a solution place if and only if $\mathcal{P}$ is an order-suitable place. In the affirmative case, $\mathcal{P}$ contains exactly n non-constant solutions and they have ramification index equals n.

Critical Points

Let $(c_0, c_1) \in \mathcal{C}(F)$ and let $(a(t), b(t)) \in \textbf{LocalPar}(\mathcal{C}(F))$ be centered at (c_0, c_1). If F is square-free, there are only finitely many such curve points (c_0, c_1) where $(a(t), b(t))$ is not order-suitable with $n = 1$.

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Example 1

For $F = y'^2 - y^3 - y^2$ the critical points are $(-1, 0), (0, 0)$.

Summary

Algorithm arising from the proof

Given $F \in \mathbb{K}[y, y']$ square-free.

1) Compute a generic power series solution.
2) Compute the critical points $(y_0, p_0) \in \mathcal{C}(F) \cap \mathcal{C}(\frac{\partial F}{\partial y'})$.
3) For every critical point compute a representative $(a(t), b(t))$ of every place at (y_0, p_0) and determine n.
4) Take $s(t) = s_1 t + s_2 t^2 + \cdots$ with s_i undetermined and compute them from

$$a'(s(t))\, s'(t) = n\, t^{n-1}\, b(s(t)). \tag{3}$$

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$$a'(s(t))\, s'(t) = n\, t^{n-1}\, b(s(t)). \tag{3}$$

Equation 3 is called the associated differential equation and can be solved for example with the Newton-Puiseux method for differential equations. Note that in every step we can ensure convergence.

Remarks

Theorem [Convergence]

Let $\mathbb{K} = \mathbb{C}$. Then, all formal Puiseux series solutions of $F(y, y') = 0$ are convergent.

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Let $\mathbb{K} = \mathbb{C}$. Then, all formal Puiseux series solutions of $F(y, y') = 0$ are convergent.

Moreover, all the coefficients of $s(t)$ are in the same field as the coefficients of $a(t)$ and $b(t)$. In this way, we can also say something about the field extensions in which the coefficients of the solutions are.

Example 1

For $F = y'^2 - y^3 - y^2$, the local parametrizations at $(0,0)$, namely

$$(t, \pm t - t^2/2 + \mathcal{O}(t^3)),$$

are not order-suitable, whereas that one at $(-1,0)$,

$$(t^2 - 1, t - t^3)$$

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is order-suitable with $n = 1$: The associated differential equation

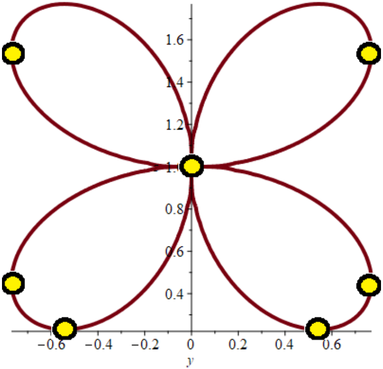
$$(a(s(t)))' = 2s(t)s'(t) = b(s(t)) = s(t) - s(t)^3$$

leads to $s(t) = t/2 - t^3/24 + t^5/240 + \mathcal{O}(t^7)$ and therefore,

$$a(s(x)) = -1 + x^2/4 - x^4/24 + \mathcal{O}(x^6).$$

Example

Consider $F(y, y') = ((y' - 1)^2 + y^2)^3 - 4(y' - 1)^2 y^2 = 0.$



Then the critical points are

$$\begin{aligned}\mathcal{B} &= \{(0,1)\} \cup \left\{(\alpha,0) \,|\, \alpha^6 + 3\alpha^4 - \alpha^2 + 1 = 0\right\} \cup \\ &\left\{\left(\tfrac{4\beta}{9}, \gamma\right) \,|\, \beta^2 = 3, 27\gamma^2 - 54\gamma + 19 = 0\right\} \cup \{\infty, \infty\}\end{aligned}$$

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- At $\mathbf{c}_1 = (0,1)$ there are 4 places defined by

$$\begin{array}{ll}(a,b) = (t^2, 1 + \sqrt{2}t - \frac{3t^2}{4\sqrt{2}} + \mathcal{O}(t^3)) & \text{suitable with } n = 2 \\ (-t^2, 1 - \sqrt{2}t - \frac{3t^2}{4\sqrt{2}} + \mathcal{O}(t^3)) & \text{suitable with } n = 2 \\ (t, 1 + \frac{t^2}{2} + \frac{3t^4}{16} + \mathcal{O}(t^6)) & \text{suitable with } n = 1 \\ (t, 1 - \frac{t^2}{2} - \frac{3t^4}{16} + \mathcal{O}(t^6)) & \text{suitable with } n = 1\end{array}$$

For $(a(t), b(t))$ the associated differential equation is

$$s(t)\, s'(t) = t\, \left(1 + \sqrt{2}S(t) - \frac{3S(t)^2}{4\sqrt{2}}\right)$$

with the solutions

$$\begin{aligned} s_1(t) &= t + \tfrac{\sqrt{2}t^2}{3} - \tfrac{t^3}{18} + \mathcal{O}(t^4), \\ s_2(t) &= -t + \tfrac{\sqrt{2}t^2}{3} + \tfrac{t^3}{18} + \mathcal{O}(t^4). \end{aligned}$$

By considering all places at $\mathbf{c}_1$ we obtain

$$\left\{ \begin{array}{l} a(s_1(x^{1/2})) = x + \frac{2\sqrt{2}x^{3/2}}{3} + \frac{x^2}{3} + \mathcal{O}(x^{5/2}), \\ a(s_2(x^{1/2})) = x - \frac{2\sqrt{2}x^{3/2}}{3} + \frac{x^2}{3} + \mathcal{O}(x^{5/2}), \\ x + \frac{2\sqrt{2}ix^{3/2}}{3} - \frac{x^2}{3} + \mathcal{O}(x^{5/2}),\ x - \frac{2\sqrt{2}ix^{3/2}}{3} - \frac{x^2}{3} + \mathcal{O}(x^{5/2}), \\ x + \frac{x^3}{6} + \frac{17x^5}{240} + \mathcal{O}(x^6),\ x - \frac{x^3}{6} + \frac{17x^5}{240} + \mathcal{O}(x^6) \end{array} \right\}$$

- For $\mathbf{c}_\alpha = (\alpha, 0)$ with $\alpha^6 + 3\alpha^4 - \alpha^2 + 1 = 0$ we get the places

$$\left(\alpha + t, \left(\frac{11}{19}\alpha^5 + \frac{36}{19}\alpha^3 + \frac{4}{19}\alpha\right) t + \mathcal{O}(t^2)\right),$$

  which are not suitable.
- Let $\mathbf{c}_{\beta,\gamma} = \left(\frac{4\beta}{9}, \gamma\right)$, where $\beta^2 = 3$, and $27\gamma^2 - 54\gamma + 19 = 0$. Then the places

$$\left(\frac{4\beta}{9} + t^2, \gamma + \frac{\sqrt{\beta}i}{\sqrt{3}} t + \mathcal{O}(t^2)\right)$$

  are suitable with $n = 2$.

- Let us consider the curve point $(\infty, \infty) \in \mathcal{C}(F)$. We do this by considering instead $\mathbf{c}_\infty = (0, 0) \in \mathcal{C}(num(F(1/y, -y'/y^2)))$. We obtain the places

$$(t^3, \pm it^3 + \mathcal{O}(t^4)),$$

which are not suitable.