# The Algebro-Geometric Method for Solving Algebraic Differential Equations 

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## Abstract

We present the algebro-geometric method for computing explicit formula solutions for algebraic differential equations (ADEs).

An algebraic differential equation is a polynomial relation between a function, some of its partial derivatives, and the variables in which the function is defined.

Regarding all these quantities as unrelated variables, we get an algebraic hypersurface on which the solution is to be found. Parametrizations of this corresponding hypersurface are closely related to solutions of the ADE.

This approach is relatively well understood for rational and algebraic solutions of single algebraic ordinary differential equations (AODEs). First steps are taken in a generalization to other types of solutions and to partial differential equations.

## Outline

Symbolic solutions of ADEs - The problem
Rational parametrizations
Rational solutions of AODEs of order 1

Algebraic solutions of AODEs
Classification of AODEs / differential orbits
Extending the algebro-geometric method
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## Symbolic solutions of ADEs - The problem

An algebraic differential equation (ADE) is given by a polynomial relation between a desired function $y\left(x_{1}, \ldots, x_{m}\right)$, some of its derivatives, and the variables $x_{1}, \ldots, x_{m}$ :

$$
\begin{equation*}
F\left(x_{1}, \ldots, x_{m}, y, \ldots, \frac{\partial y}{\partial x_{i}}, \ldots, \frac{\partial^{k} y}{\partial x_{j_{1}}, \ldots, x_{j_{k}}}, \ldots\right)=0 \tag{1}
\end{equation*}
$$

$m=1$ : algebraic ordinary differential equation (AODE) $m>1$ : algebraic partial differential equation (APDE)

An algebraic ordinary differential equation (AODE) is given by

$$
F\left(x, y, y^{\prime}, \ldots, y^{(n)}\right)=0
$$

where $F$ is a differential polynomial in $K[x]\{y\}$ with $K$ being a differential field and the derivation ' being $\frac{d}{d x}$.
Such an AODE is autonomous iff $F \in K\{y\}$.
The radical differential ideal $\{F\}$ can be decomposed

$$
\{F\}=\underbrace{(\{F\}: S)}_{\text {general component }} \cap \underbrace{\{F, S\}}_{\text {singular component }},
$$

where $S$ is the separant of $F$ (derivative of $F$ w.r.t. $y^{(n)}$ ). If $F$ is irreducible, $\{F\}: S$ is a prime differential ideal; its generic zero is called a general solution of the AODE $F\left(x, y, y^{\prime}, \ldots, y^{(n)}\right)=0$.
J.F. Ritt, Differential Algebra (1950)

## Problem: Rational general solution of AODE of order 1

given: an AODE $F\left(x, y, y^{\prime}\right)=0, F$ irreducible in $\overline{\mathbb{Q}}\left[x, y, y^{\prime}\right]$
decide: does this AODE have a rational general solution
find: if so, find it

Example: Consider the differential equation

$$
F\left(x, y, y^{\prime}\right) \equiv y^{\prime 2}+3 y^{\prime}-2 y-3 x=0
$$

general solution: $y=\frac{1}{2}\left((x+c)^{2}+3 c\right)$, where $c$ is an arbitrary constant.
The separant of $F$ is $S=2 y^{\prime}+3$. So the singular solution of $F$ is $y=-\frac{3}{2} x-\frac{9}{8}$.

## Rational parametrizations

An algebraic variety $\mathcal{V}$ is the zero locus of a (finite) set of polynomials $F$, or of the ideal $I=\langle F\rangle$.
A rational parametrization of $\mathcal{V}$ is a rational map $\mathcal{P}$ from a full (affine, projective) space covering $\mathcal{V}$; i.e. $\mathcal{V}=\overline{\operatorname{im}(\mathcal{P})}$ (Zariski closure).
A variety having a rational parametrization is called unirational; and rational if $\mathcal{P}$ has a rational inverse.
other types of parametrizations:
radical parametrization
local (power series) parametrization

Example: The singular cubic

$$
y^{2}-x^{3}-x^{2}=0
$$

has the rational, in fact polynomial, parametrization

$$
x(t)=t^{2}-1, \quad y(t)=t^{3}-t
$$

So this is a unirational curve. It is in fact rational: $t=y / x$.


- a parametrization of a variety is a generic point or generic zero of the variety; i.e. a polynomial vanishes on the variety if and only if it vanishes on this generic point
- so only irreducible varieties can be rational
- a rationally invertible parametrization $\mathcal{P}$ is called a proper parametrization; every rational curve or surface has a proper parametrization (Lüroth, Castelnuovo); but not so in higher dimensions

For details on parametrizations of algebraic curves we refer to J.R. Sendra, F. Winkler, S. Pérez-Díaz, Rational Algebraic Curves - A Computer Algebra Approach, Springer-Verlag Heidelberg (2008)

## Rational solutions of AODEs of order 1

The autonomous case $F\left(y, y^{\prime}\right)=0$
This is the case described by Feng\&Gao $(2004,2006)$ based on our (Sendra\&W.) degree bounds for rational parametrizations.
First one concentrates on algebraic and geometric questions:

- A rational solution of $F\left(y, y^{\prime}\right)=0$ corresponds to a proper (because of the degree bounds) rational parametrization of the algebraic curve $F(y, z)=0$.
- Conversely, from a proper rational parametrization $(f(x), g(x))$ of the curve $F(y, z)=0$ we get a rational solution of $F\left(y, y^{\prime}\right)=0$ if and only if there is a linear rational function $T(x)$ such that $f(T(x))^{\prime}=g(T(x))$.

Example: Consider the autonomous order-1 AODE

$$
F\left(y, y^{\prime}\right)=\left(y^{\prime}\right)^{2}-y^{3}-y^{2}=0
$$

To this differential equation we associate the algebraic equation $F(y, z)=0$. This algebraic equation defines the singular cubic, which is parametrized by $(r(t), s(t))=\left(t^{2}-1, t^{3}-t\right)$.
There is no linear transformation such that the second component becomes the derivative of the first. So this AODE has no general rational solution. However, it does have the singular rational solutions $y=0$ or $y=-1$.

Example: Consider the autonomous AODE

$$
F\left(y, y^{\prime}\right)=\left(y^{\prime}\right)^{3}-27 y^{2}-54 y-27
$$

The associated algebraic curve is rationally parametrized by $\mathcal{P}(t)=(r(t), s(t))$, where

$$
r(t)=\frac{19 t^{3}-12 t^{2}-6 t-1}{(2 t+1)^{3}}, \quad s(t)=\frac{27 t^{2}}{(2 t+1)^{2}} .
$$

But this curve is also parametrized by $\tilde{\mathcal{P}}(t)=(\tilde{r}(t), \tilde{s}(t))$, where

$$
\tilde{r}(t)=t^{3}+\frac{9}{2} t^{2}+\frac{27}{4} t+\frac{19}{8}, \quad \tilde{s}(t)=\frac{3}{4}(2 t+3)^{2}
$$

so that the second component is the derivative of the first. Indeed, $s(t) / r^{\prime}(t)=a(t-b)^{2}=4 / 3(t+1 / 2)^{2}$. The Möbius transformation $t \mapsto(a b t-1) /(a t)$ maps $\mathcal{P}$ to $\tilde{\mathcal{P}}$. So a rational solution is

$$
y(x)=r\left(\frac{a b x-1}{a x}\right)=x^{3}+\frac{9}{2} x^{2}+\frac{27}{4} x+\frac{19}{8}
$$

and a general rational solution is $\hat{y}=y(x+c)$.

Example: (a) $y^{\prime}=y$.

$$
\mathcal{P}(x)=(r(x), s(x))=(x, x)
$$

There is no linear transformation which results in $r^{\prime}=s$, so there is no rational solution.
(b) $y^{\prime}=y^{2}$.

$$
\mathcal{P}(x)=(r(x), s(x))=\left(x, x^{2}\right)
$$

The linear substitution $x \rightarrow-(1 / x)$ results in the parametrization

$$
\begin{gathered}
\mathcal{Q}(x)=\left(-\frac{1}{x}, \frac{1}{x^{2}}\right) \\
\text { So } y=\frac{-1}{x} \quad \text { is a rational solution } \\
\text { and } \quad \hat{y}=\frac{-1}{x+c} \quad \text { is a general rational solution }
\end{gathered}
$$

The general (non-autonomous) case $F\left(x, y, y^{\prime}\right)=0$
L.X.C. Ngô, F.W., JSC, (2010, 2011)

It is now natural to assume that the solution surface $F(x, y, z)=0$ is a rational algebraic surface, i.e. rationally parametrized by

$$
\mathcal{P}(s, t)=\left(\chi_{1}(s, t), \chi_{2}(s, t), \chi_{3}(s, t)\right) .
$$

Then $\mathcal{P}(s, t)$ creates a rational solution of $F\left(x, y, y^{\prime}\right)=0$ if and only if we can find two rational functions $s(x)$ and $t(x)$ which solve the following associated system (a planar rational system):

$$
\begin{equation*}
s^{\prime}=\frac{f_{1}(s, t)}{g(s, t)}, \quad t^{\prime}=\frac{f_{2}(s, t)}{g(s, t)}, \tag{2}
\end{equation*}
$$

where $f_{1}(s, t), f_{2}(s, t), g(s, t)$ are rational functions in $s, t$.

Theorem
There is a one-to-one correspondence between rational general solutions of the AODE $F\left(x, y, y^{\prime}\right)=0$ and rational general solutions of its associated system.

The associated system is

- autonomous
- of order 1
- of degree 1 in the derivatives of the parameters

Every non-trivial rational solution $\mathcal{R}(x)=(s(x), t(x))$ of the associated system implicitly defines a curve $G(s, t)$ s.t.
$G(s(x), t(x))=0$.
By differentiation and taking into account (2), we get that $G_{s} \cdot f_{1}+G_{t} \cdot f_{2}$ vanishes at $\mathcal{R}(x)$; so $G_{s} \cdot f_{1}+G_{t} \cdot f_{2} \in\langle G\rangle$. Such curves $G(s, t)$ are called invariant algebraic curves. The irreducible factors of an invariant curve are also invariant curves.

## Lemma

Every non-trivial rational solution of the associated system corresponds to a rational invariant algebraic curve, i.e. a curve $G(s, t)=0$ satisfying $G_{s} \cdot f_{1}+G_{t} \cdot f_{2} \in\langle G\rangle$.

In the generic case (assoc. system has no dicritical points) there is an upper bound for irreducible invariant algebraic curves.

Rational invariant algebraic curves create candidates for rational solutions of the associated system, and therefore of the original AODE;
if we can find a linear transformation s.t. the derivation of the first component is equal to the second.

Example: Consider the differential equation

$$
F\left(x, y, y^{\prime}\right) \equiv y^{\prime 2}+3 y^{\prime}-2 y-3 x=0
$$

The solution surface $z^{2}+3 z-2 y-3 x=0$ has the parametrization

$$
\mathcal{P}(s, t)=\left(\frac{t}{s}+\frac{2 s+t^{2}}{s^{2}},-\frac{1}{s}-\frac{2 s+t^{2}}{s^{2}}, \frac{t}{s}\right) .
$$

This is a proper parametrization and its associated system is

$$
s^{\prime}=s t, \quad t^{\prime}=s+t^{2}
$$

Irreducible invariant algebraic curves of the system are:

$$
G(s, t)=s, \quad G(s, t)=t^{2}+2 s, \quad G(s, t)=s^{2}+c t^{2}+2 c s
$$

The third algebraic curve $s^{2}+c t^{2}+2 c s=0$ depends on a transcendental parameter $c$. It can be parametrized by

$$
\mathcal{Q}(x)=\left(-\frac{2 c}{1+c x^{2}},-\frac{2 c x}{1+c x^{2}}\right) .
$$

Running Step 5 in RATSOLVE, the differential equation defining the reparametrization is $T^{\prime}=1$. Hence $T(x)=x$. So the rational solution in this case is

$$
s(x)=-\frac{2 c}{1+c x^{2}}, \quad t(x)=-\frac{2 c x}{1+c x^{2}}
$$

Since $G(s, t)$ contains a transcendental constant, the above solution is a rational general solution of the associated system.
Therefore, the rational general solution of $F\left(x, y, y^{\prime}\right)=0$ is

$$
y=\frac{1}{2} x^{2}+\frac{1}{c} x+\frac{1}{2 c^{2}}+\frac{3}{2 c},
$$

which, after a change of parameter, can be written as

$$
y=\frac{1}{2}\left(x^{2}+2 c x+c^{2}+3 c\right)
$$

Recently we have been able to derive an algorithm for deciding the existence of a strong rational general solution, i.e. a general solution in $K(c, x)$ (also rational in the constant $c$ ). Instead of parametrizing a surface we parametrize a curve over a more complicated field:

$$
\mathcal{C}=\left\{\left(a_{1}, a_{2}\right) \in \mathbb{A}^{2}(\overline{K(x)}) \mid F\left(x, a_{1}, a_{2}\right)=0\right\}
$$

All the solutions of examples above were strong rational solutions.

If $F=0$ has a strong rational general solution, then it has an associated Riccati equation

$$
\omega^{\prime}=a_{0}(x)+a_{1}(x) \omega+a_{2}(x) \omega^{2}
$$

Existence of rational general solution can be decided.
So we have a full decision algorithm for finding rational general solutions of strongly parametrizable AODEs.
Almost all first-order AODEs in the collection of Kamke are in this class. These cover 64 percent of all the first-order ODEs in Kamke.

Example 1.537 in Kamke: Consider the differential equation

$$
F\left(x, y, y^{\prime}\right)=\left(x y^{\prime}-y\right)^{3}+x^{6} y^{\prime}-2 x^{5} y=0
$$

The associated curve defined by $F(x, y, z)=0$ can be parametrized by

$$
\mathcal{P}(t)=\left(-\frac{t^{3} x^{5}-t^{2} x^{6}+(t-x)^{3}}{t^{3} x^{5}},-\frac{2 t^{3} x^{5}-2 t^{2} x^{6}+(t-x)^{3}}{t^{3} x^{6}}\right)
$$

Therefore, the associated differential (Riccati) equation with respect to $\mathcal{P}$ is

$$
\omega^{\prime}=\frac{1}{x^{2}} \omega(2 \omega-x)
$$

This Riccati equation has the rational general solution

$$
\omega(x)=\frac{x}{1+c x^{2}}
$$

Hence, the original AODE $F\left(x, y, y^{\prime}\right)=0$ has the strong rational general solution

$$
y(x)=c x\left(x+c^{2}\right) .
$$

## Algebraic solutions of AODEs

Aroca et al.: algebraic general solutions of autonomous AODEs
Vo, W.: algebraic general solutions of (non-autonomous) AODEs given a bound for the algebraic extension degree

Example: Consider the differential equation

$$
4 x(x-y) y^{\prime 2}+2 x y y^{\prime}-5 x^{2}+4 x y-y^{2}=0
$$

The solution surface is parametrized by the rational map

$$
\mathcal{P}(s, t):=\left(s,-\frac{t^{2}-5 t s+5 s^{2}}{s}, \frac{t^{2}-4 s t+5 s^{2}}{2 s(t-2 s)}\right)
$$

The associated system with respect to $\mathcal{P}$ is

$$
s^{\prime}=1, \quad t^{\prime}=\frac{t^{2}-3 s^{2}}{2 s(t-2 s)}
$$

The associated system has the rational first integral $W=\frac{t^{2}-4 s t+3 s^{2}}{s}$.
Thus it has an algebraic solution $(s(x), t(x)):=(x, \bar{t}(x, c))$, where $\bar{t}(x, c)$ is a root of the algebraic equation $t^{2}-4 x t+3 x^{2}-c x=0$. The solution of the associated system can be transformed into a solution of the original equation, and we get that

$$
y(x)=\frac{\sqrt{c x(c x+1)}-1}{c}
$$

is an algebraic general solution of the differential equation.

## Classification of AODEs / differential orbits

- consider a group of transformations leaving the associated system of an AODE invariant; orbits w.r.t. such a transformation group contain AODEs of equal complexity in terms of determining rational solutions
- we have studied affine and birational transformations
- it turns out that being autonomous is not an intrinsic property of an AODE; certain classes contain both autonomous and non-autonomous AODEs


## Affine transformations

Ngô, Sendra, W., Cont.Math. 572 (2012)
The group $\mathcal{G}$ of affine transformations

$$
\begin{aligned}
L: \mathbb{A}^{3}(\mathbb{K}) & \longrightarrow \\
& \longrightarrow\left(\begin{array}{lll} 
& \\
& (\mathbb{K}) \\
v & \mapsto\left(\begin{array}{lll}
1 & 0 & 0 \\
b & a & 0 \\
0 & 0 & a
\end{array}\right) v+\left(\begin{array}{l}
0 \\
c \\
b
\end{array}\right)
\end{array}\right) .
\end{aligned}
$$

leaves the associated system of an AODE invariant, and therefore also the rational solvability.
We call such a transformation solution preserving.

## Theorem

The group $\mathcal{G}$ defines a group action on AODEs by

$$
\begin{aligned}
\mathcal{G} \times \mathcal{A O D E} & \rightarrow \mathcal{A O D E} \\
(L, F) & \mapsto L \cdot F=\left(F \circ L^{-1}\right)\left(x, y, y^{\prime}\right) .
\end{aligned}
$$

Theorem
Let $F$ be a parametrizable $A O D E$, and $L \in \mathcal{G}$. For every proper rational parametrization $\mathcal{P}$ of the surface $F(x, y, z)=0$, the associated system of $F\left(x, y, y^{\prime}\right)=0$ w.r.t. $\mathcal{P}$ and the associated system of $(L \cdot F)\left(x, y, y^{\prime}\right)=0$ w.r.t. $L \circ \mathcal{P}$ are equal.

Example: As in the previous example we consider the differential equation

$$
F\left(x, y, y^{\prime}\right) \equiv y^{\prime 2}+3 y^{\prime}-2 y-3 x=0
$$

We first check whether in the class of $F$ there exists an autonomous AODE. For this, we apply a generic $L$ to $F$ to get
$(L \cdot F)\left(x, y, y^{\prime}\right)=\frac{1}{a^{2}} y^{\prime 2}+\frac{3}{a} y^{\prime}-\frac{2 b}{a^{2}} y^{\prime}-\frac{2}{a} y+\frac{2 b}{a} x-3 x-\frac{3 b}{a}+\frac{b^{2}}{a^{2}}+\frac{2 c}{a}$.
Therefore, for every $a \neq 0$ and $b$ such that $2 b-3 a=0$, we get an autonomous AODE. In particular, for $a=1, b=3 / 2$, and $c=0$ we get

$$
L=\left[\left(\begin{array}{lll}
1 & 0 & 0 \\
\frac{3}{2} & 1 & 0 \\
0 & 0 & 1
\end{array}\right),\left(\begin{array}{l}
0 \\
0 \\
3 \\
\frac{2}{2}
\end{array}\right)\right]
$$

i.e., we obtain

$$
F\left(L^{-1}\left(x, y, y^{\prime}\right)\right) \equiv y^{\prime 2}-2 y-\frac{9}{4}=0
$$

## Birational transformations

The group $\mathcal{G}$ of birational transformations from $\mathbb{K}^{3}$ to $\mathbb{K}^{3}$ of the form

$$
\begin{aligned}
& \Phi\left(u_{1}, u_{2}, u_{3}\right)= \\
& \left(u_{1}, \quad \frac{a u_{2}+b}{c u_{2}+d}, \quad \frac{\partial}{\partial u_{1}}\left(\frac{a u_{2}+b}{c u_{2}+d}\right)+\frac{\partial}{\partial u_{2}}\left(\frac{a u_{2}+b}{c u_{2}+d}\right) \cdot u_{3}\right),
\end{aligned}
$$

where $a, b, c, d \in \mathbb{K}\left[u_{1}\right]$ such that $a d-b c \neq 0$, defines a group action on $\mathcal{A O D E}$ by

$$
\Phi \cdot F=\left(F \circ \Phi^{-1}\right)\left(x, y, y^{\prime}\right) .
$$

These birational transformations leave the associated system of an AODE invariant, and therefore also the rational solvability.

## Problem:

given: $F\left(x, y, y^{\prime}\right) \in \mathcal{A O D E}$,
decice: does there exist a solution preserving transformation $\Phi$ s.t. $G=\Phi \cdot F$ is autonomous?
And, if so, can we compute such a $\Phi$ (and therefore $G$ ) ?
this is an open problem!

Example: Consider the first order AODE

$$
\begin{aligned}
F\left(x, y, y^{\prime}\right)= & 25 x^{2} y^{\prime 2}-50 x y y^{\prime}+25 y^{2}+12 y^{4}-76 x y^{3}+ \\
& 168 x^{2} y^{2}-144 x^{3} y+32 x^{4}=0 .
\end{aligned}
$$

Using the transformation

$$
\Phi(u, v, w)=\left(u, \frac{u-3 v}{-2 u+v}, \frac{-5 v}{(2 u-v)^{2}}+\frac{5 u}{(2 u-v)^{2}} w\right)
$$

we get the autonomous equation

$$
G\left(y, y^{\prime}\right)=F\left(\Phi^{-1}\left(x, y, y^{\prime}\right)\right)=y^{\prime 2}-4 y=0
$$

Observe that $F$ cannot be transformed into an autonomous AODE by affine transformations.
The rational general solution $y=(x+c)^{2}$ of $G\left(y, y^{\prime}\right)=0$ is transformed into the rational general solution of $F\left(x, y, y^{\prime}\right)=0$ :

$$
y=\frac{x\left(2(x+c)^{2}+1\right)}{(x+c)^{2}+3}
$$

## Extending the algebro-geometric method

(a) $y^{8} y^{\prime}-y^{5}-y^{\prime}=0$ :
parametrization: $\left(\frac{1}{t}, \frac{t^{3}}{1-t^{8}}\right)$,
radical solution: $y(x)=-\left(2(x+c)-\sqrt{-1+4(x+c)^{2}}\right)^{-1 / 4}$
(b) $4 y^{7}-4 y^{5}-y^{3}-2 y^{\prime}-8 y^{2} y^{\prime}+8 y^{4} y^{\prime}+8 y y^{\prime 2}=0:($ genus 1$)$
parametrization: $\left(\frac{1}{t}, \frac{-4+4 t^{2}+t^{4}}{t\left(4 t^{2}-4 t^{4}-t^{6}-\sqrt{t^{12}+8 t^{10}+16 t^{8}-16 t^{4}}\right)}\right)$
radical solution: $y(x)=-\frac{\sqrt{1+c+x}}{\sqrt{1+(c+x)^{2}}}$
(c) $y^{3}+y^{2}+y^{\prime 2}=0$ :
parametrization: $\left(-1-t^{2}, t\left(-1-t^{2}\right)\right)$,
trigonometric solution: $y(x)=-1-\tan \left(\frac{x+c}{2}\right)^{2}$
(d) $y^{2}+y^{\prime 2}+2 y y^{\prime}+y=0$ :
parametrization: $\left(-\frac{1}{(1+t)^{2}},-\frac{t}{(1+t)^{2}}\right)$
exponential solution: $y(x)=-e^{-x}\left(-1+e^{x / 2}\right)^{2}$
first steps taken

- radical solutions
- power series solutions
- AODEs of higher order
- systems of APDEs


## Conclusion

- we can decide whether an AODE has a strong general rational solution, and if it has, we can determine such a general rational solution
- we have a generalization of the solution method for rational solutions to other types of solutions; but not yet decision procedures
- we have taken first steps towards extending the method to algebraic partial differential equations


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## Thank you for your attention!



