

## **Chapter 5**

### **Rational general solutions of AODEs of order 1**

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# The Algebro-Geometric Method for Solving Algebraic Differential Equations – A Survey

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**Abstract** We present the algebro-geometric method for computing explicit formula solutions for algebraic differential equations (ADEs). An algebraic differential equation is a polynomial relation between a function, some of its partial derivatives, and the variables in which the function is defined. Regarding all these quantities as unrelated variables, the polynomial relation leads to an algebraic relation defining a hypersurface on which the solution is to be found. A solution in a certain class of functions, such as rational or algebraic functions, determines a parametrization of the hypersurface in this class. So in the algebro-geometric method we first decide whether a given ADE can be parametrized with functions from a given class; and in the second step we try to transform a parametrization into one respecting also the differential conditions. This approach is relatively well understood for rational and algebraic solutions of single algebraic ordinary differential equations (AODEs). First steps are taken in a generalization to systems and to partial differential equations.

**Keywords** algebraic differential equation, exact solution, parametrization of curves

**MR(2000) Subject Classification** 12H05, 14M20, 34A05, 34A26, 68W30

## 1 Introduction

In this survey we are concerned with an algebraic and geometric method for determining symbolic solutions to *algebraic differential equations (ADEs)*. An ADE is a polynomial relation between a function  $y(x_1, \dots, x_n)$ , some of its derivatives and the variables  $x_1, \dots, x_n$ ,

$$F(x_1, \dots, x_n, y, \dots, \frac{\partial y}{\partial x_i}, \dots, \frac{\partial^k y}{\partial x_{j_1} \dots \partial x_{j_k}}, \dots) = 0. \quad (1)$$

Of course, only finitely many derivatives can actually appear in the polynomial  $F$ . The highest  $k$  appearing in (1) is the *order* of the ADE  $F = 0$ . If the total degree

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of the polynomial  $F$  in  $y$  and its derivatives is 1, we have a linear differential equation. Here we are mainly concerned with non-linear ADEs of order 1.

In case  $n = 1$  we speak of an *algebraic ordinary differential equation (AODE)*; so there is a (differential) polynomial  $F \in K(x)\{y\}$  such that the AODE can be written as

$$F(x, y, y', \dots, y^{(n)}) = 0 . \quad (2)$$

$F$  defines the differential equation. Without loss of generality we may consider  $F$  to be irreducible. For otherwise we may factor  $F$  and solve the AODEs defined by the factors. An AODE is *autonomous* iff it is independent of  $x$ ; i.e., it is of the form

$$F(y, y', \dots, y^{(n)}) = 0 , \quad (3)$$

with  $F \in K\{y\}$ . If  $n > 1$  we speak of an *algebraic partial differential equation (APDE)*.

We take a computer algebra point of view; i.e. we typically want to compute solution formulas, not simply numerical values at particular points. For an introduction to computer algebra in general, and to the algorithmic basis of this project in particular, we refer to [44].

Differential algebra, and in particular differential computer algebra, is investigating and modeling differential equations in several ways: by differential polynomials, by linear differential operators, and also by differential Galois theory. Already in the early decades of the 20th century Janet [19] worked on an elimination theory for partial differential equations. The theory of differential polynomials was introduced by Ritt [34] and has been vigorously developed since then. Classical Galois theory, i.e., the theory of automorphism groups of solutions to algebraic equations, is gradually being extended and applied to the solutions of differentials equations, e.g. in [21], [24] and [39]. Lie symmetries can often be used for determining solutions; cf. [35]. AODEs are typically non-linear. There are still no general solution algorithms for such problems. Special cases are treated, e.g., in [22], [38], and [3], [4]. The solution methods can also be distinguished w.r.t. the type of solution they are looking for; e.g., rational, algebraic, or Liouvillian solutions.

First-order AODEs have been studied a lot and there is a variety of solution methods for special classes of such ODEs. The study of AODEs can be dated back to the work of Fuchs [11], and Poincaré [33]. In [25], Malmquist studied the class of first-order AODEs having transcendental meromorphic solutions, and Eremenko revisited later in [7]. By using the result of Matsuda [26] on classification of differential algebraic function fields without movable critical points, Eremenko presented an implicit characterization of a degree bound for rational

solutions [8]. In [22], Kovacic solved completely the problem of computing Liouvillian solutions of second order linear AODEs with rational function coefficients. Kovacic also proposed an algorithm for determining all rational solutions of a Riccati equation. Hubert [16] found implicit solutions by computing Gröbner bases.

A new algebro-geometric approach was first introduced by Feng and Gao. In [9, 10] they provided a polynomial time algorithm for computing rational general solutions of first-order autonomous AODEs. In [6], Chen and Ma combined an algebro-geometric method with Fuchs' theorem about first-order AODEs without movable critical point and studied a special kind of rational general solutions. In [30–32], Ngô and Winkler extended the algebro-geometric method to the class of non-autonomous parametrizable first-order AODEs and studied their rational general solutions. Following this direction, a generalization to the class of higher order AODEs [18], and even to algebraic partial differential equations [12, 13] has been proposed. Whereas for general first-order AODEs we still do not have a decision algorithm for the existence of rational general solutions, we can now decide whether such an AODE has a so-called strong rational general solution; i.e., a rational general solution in which the transcendental constant appears rationally. This has recently been shown in [41]. In [42] we compute all rational solutions of a wide class of first-order AODEs. A practical application of the algebro-geometric method can be found in [14], where explicit solutions of AODEs give rise to formulas for Zolotarev polynomials.

Let  $K$  be an algebraically closed differential field, with derivation  $\partial$ . For brevity, often we write  $'$  instead of  $\partial$ . The derivation is extended to  $K(x)$ , the field of rational functions, by  $x' = 1$ . Furthermore we extend the derivation to differential polynomials in the differential variable  $y$  over  $K(x)$ . So we consider variables  $y^{(i)}$ ,  $i \in \mathbb{N}$ , with  $y = y^{(0)}$  and for every  $i \in \mathbb{N}$  we have  $\partial(y^{(i)}) = y^{(i+1)}$ . The (differential) ring of differential polynomials is denoted by  $K(x)\{y\}$ . An ideal  $I$  in  $K(x)\{y\}$  is a *differential ideal* if the derivation does not lead out of  $I$ ; i.e., if  $F \in I$  then also  $F' \in I$ .

According to Ritt [34], Chap. 2, the radical differential ideal  $\{F\}$  generated by the irreducible differential polynomial  $F$ ,  $F \in K(x)\{y\}$ , can be decomposed as

$$\{F\} = \underbrace{(\{F\} : S)}_{\text{general component}} \cap \underbrace{\{F, S\}}_{\text{singular component}},$$

where  $S$  is the separant of  $F$ ; i.e., the derivative of  $F$  w.r.t.  $y^{(n)}$ ,  $n$  being the order of  $F$ . The general component  $\{F\} : S$  encodes the conditions  $F = 0, S \neq 0$ . It is a prime differential ideal, so it has a generic zero. The singular component

$\{F, S\}$  encodes the conditions  $F = 0, S = 0$ .

A function  $y(x)$  in an extension field of  $K(x)$  satisfying  $F(x, y, y', \dots, y^{(n)}) = 0$  is called a *solution* of the AODE (2). A *rational solution* is a solution in  $K(x)$ . A solution of the general component is called a *regular solution*, a solution of the singular component is called a *singular solution*.

Consider  $c$  transcendental over  $K$ . A solution  $y(c) \in \overline{K(c)}(x) \setminus K(x)$  of (2) is called a *general rational solution*. A general rational solution is a generic point of the general component  $\{F\} : S$ ; i.e., the membership problem for  $\{F\} : S$  can be decided by substituting  $y(c)$ . Compare [34], Chap. 2.

## 2 Rational solutions of AODEs

The study of first-order algebraic ODEs can be dated back to the work of Fuchs [11] and Poincaré [33]. Malmquist studied the class of first-order AODEs having transcendental meromorphic solutions in [25], and later Eremenko revisited this problem. By using the result of Matsuda [26] on classification of differential algebraic function fields without movable critical points, Eremenko [8] presented a theoretical consideration on a degree bound of rational solutions. The problem of finding closed form solutions of first-order AODEs has been considered in several papers. Kovacic [22] solved completely the problem of computing Liouvillian solutions of a second-order linear ODE with rational function coefficients. He also proposed an algorithm for determining all rational solutions of a Riccati equation. Hubert [16] found implicit solutions by computing Gröbner bases.

In this section we deal with the problem of deciding the existence of rational solutions of AODEs of order 1,

$$F(x, y, y') = 0 \tag{4}$$

over an algebraically closed field  $K$  of characteristic 0. As the sole additional requirement we need to be able to parametrize algebraic curves (or surfaces) over  $K$ . If the AODE has rational solutions, we want to find a rational general solution; i.e., a rational solution containing a transcendental constant  $c$ .

**Example 1:** Consider the non-autonomous AODE over  $\overline{\mathbb{Q}}$

$$F \equiv y'^2 + 3y' - 2y - 3x = 0 .$$

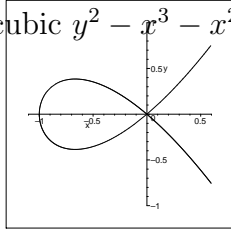
A general solution is  $y = \frac{1}{2}((x+c)^2 + 3c)$ . The separant of  $F$  is  $S = 2y' + 3$ . So the singular solution of  $F$  is  $y = -\frac{3}{2}x - \frac{9}{8}$ .  $\square$

In their seminal papers [9, 10] Feng and Gao introduced the algebro-geometric method. They related an autonomous AODE of order 1 to a plane algebraic curve, and a rational solution of the AODE to a proper rational parametrization of this

curve. So if the differential equation has a rational solution, the corresponding curve must be rationally parametrizable. Using the fact that all proper rational parametrizations of an algebraic plane curve are related by Möbius transformations and citing the strict degree bounds of parametrizations derived in [36], they finally decide whether this curve has a parametrization whose second component is the derivative of the first, thus deciding the existence of a rational solution of the AODE.

An *algebraic variety*  $\mathcal{V}$  is the zero locus of a set of (defining) polynomials, or of the ideal  $I$  generated by these polynomials. Hilbert's Basis Theorem guarantees that an algebraic variety can always be defined by finitely many polynomials. A (*rational*) *parametrization* of  $\mathcal{V}$  is a rational map  $\mathcal{P}$  from a full (affine, projective) space onto  $\mathcal{V}$ ; i.e.  $\mathcal{V} = \overline{\text{im}(\mathcal{P})}$  (Zariski closure). A variety having a rational parametrization is called *unirational*; and *rational* if  $\mathcal{P}$  has a rational inverse.

For example, the singular cubic  $y^2 - x^3 - x^2 = 0$



has the rational, in fact polynomial, parametrization  $x(t) = t^2 - 1$ ,  $y(t) = t^3 - t$ . The inverse of this parametrization is given by  $t = y/x$ . So this is a rational curve.

A parametrization of a variety is a *generic point* of the variety; i.e., a polynomial vanishes on the variety if and only if it vanishes on this generic point. Exactly the irreducible varieties have a generic point, so only irreducible varieties can be rational. An irreducible plane algebraic curve is rational if and only if it is of genus 0. A rationally invertible parametrization  $\mathcal{P}$  is called a *proper* parametrization. As a consequence of Lüroth's Theorem (see [40], Chap. 10) a plain curve can be rationally parametrized if and only if it can be properly parametrized. The same holds for surfaces in 3-space because of Castelnuovo's Theorem. But for hypersurfaces in higher dimension this is not the case, see [2]. All proper parametrizations of a plane algebraic curve are related by Möbius transformations, i.e., birational maps of degree 1 of the form  $t \mapsto (ut+v)/(wt+z)$ . For details on parametrizations of algebraic curves we refer to [37].

For a proper rational parametrization  $\mathcal{P}(t) = (r(t), s(t))$  of a plain algebraic curves we have exact degree bounds; see [36]. Based on these bounds, Feng and Gao observed that a rational solution of the autonomous order-1 AODE  $F(y, y') = 0$  corresponds to a proper rational parametrization of the algebraic curve  $F(y, z) = 0$ . Conversely, from a proper rational parametrization  $(r(t), s(t))$

of the curve  $F(y, z) = 0$  we get a rational solution of the AODE  $F(y, y') = 0$  if and only if there is a linear rational function  $L(t)$  such that  $r(L(t))' = s(L(t))$ . Such a transformation exists if and only if  $s(t)/r'(t)$  is either a constant or a quadratic polynomial with a double root  $a(t - b)^2$ . If  $L(t)$  exists, then a rational solution of  $F(y, y') = 0$  is  $y = r(L(x))$ . A rational general solution of  $F(y, y') = 0$  is then given by  $y = r(L(x + c))$ , for a transcendental constant  $c$ .

**Example 2:** (a) Consider the autonomous order-1 AODE  $F(y, y') = (y')^2 - y^3 - y^2 = 0$ . To this differential equation we associate the algebraic equation  $F(y, z) = 0$ . This algebraic equation defines the singular cubic, which is parametrized by  $(r(t), s(t)) = (t^2 - 1, t^3 - t)$ . There is no linear transformation such that the second component becomes the derivative of the first. So this AODE has no general rational solution. However, it does have the singular rational solutions  $y = 0$  or  $y = -1$ .

(b) Consider the autonomous AODE  $F(y, y') = (y')^3 - 27y^2 - 54y - 27$  of order 1. The associated algebraic curve is rationally parametrized by  $\mathcal{P}(t) = (r(t), s(t))$ , where

$$r(t) = \frac{19t^3 - 12t^2 - 6t - 1}{(2t + 1)^3}, \quad s(t) = \frac{27t^2}{(2t + 1)^2}.$$

But this curve is also parametrized by  $\tilde{\mathcal{P}}(t) = (\tilde{r}(t), \tilde{s}(t))$ , where

$$\tilde{r}(t) = t^3 + \frac{9}{2}t^2 + \frac{27}{4}t + \frac{19}{8}, \quad \tilde{s}(t) = \frac{3}{4}(2t + 3)^2,$$

so that the second component is the derivative of the first. Indeed,  $s(t)/r'(t) = a(t - b)^2 = 4/3(t + 1/2)^2$ . The Möbius transformation  $t \mapsto (abt - 1)/(at)$  maps  $\mathcal{P}$  to  $\tilde{\mathcal{P}}$ . So a rational solution is

$$y(x) = r\left(\frac{abx - 1}{ax}\right) = x^3 + \frac{9}{2}x^2 + \frac{27}{4}x + \frac{19}{8},$$

and a general rational solution is  $\hat{y} = y(x + c)$ .  $\square$

Now let us drop the assumption that the AODE is autonomous; so we are considering (4), the non-autonomous AODE of order 1. The algebro-geometric method can be applied to this situation in two different ways, which we describe in the sequel.

Approach 1: First, we could associate with the AODE  $F(x, y, y') = 0$  an algebraic surface in  $\mathbb{A}^3(\bar{K})$  defined by  $F(x, y, z) = 0$  and relate rational solutions of the former with rational parametrizations of the latter. This approach has been pursued in [30, 31].

We assume that the *solution surface*  $F(x, y, z) = 0$  is a rational algebraic surface, i.e. rationally parametrized by

$$\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t)).$$

Then  $\mathcal{P}(s, t)$  creates a rational solution of  $F(x, y, y') = 0$  if and only if we can find two rational functions  $s(x)$  and  $t(x)$  which solve the following *associated system*:

$$s' = \frac{f_1(s, t)}{g(s, t)}, \quad t' = \frac{f_2(s, t)}{g(s, t)}, \quad (5)$$

where

$$\begin{aligned} f_1(s, t) &= \frac{\partial \chi_2(s, t)}{\partial t} - \chi_3(s, t) \cdot \frac{\partial \chi_1(s, t)}{\partial t}, \\ f_2(s, t) &= \chi_3(s, t) \cdot \frac{\partial \chi_1(s, t)}{\partial s} - \frac{\partial \chi_2(s, t)}{\partial s}, \\ g(s, t) &= \frac{\partial \chi_1(s, t)}{\partial s} \cdot \frac{\partial \chi_2(s, t)}{\partial t} - \frac{\partial \chi_1(s, t)}{\partial t} \cdot \frac{\partial \chi_2(s, t)}{\partial s}. \end{aligned}$$

There is a one-to-one correspondence between rational general solutions of the AODE  $F(x, y, y') = 0$  and rational general solutions of its associated system. In particular, Theorem 3.15 in [30] states that if  $(s(x), t(x))$  is a rational general solution of the associated system, then

$$y = \chi_2(s(2x - \chi_1(s(x), t(x))), t(2x - \chi_1(s(x), t(x)))) \quad (6)$$

is a rational general solution of (4). This associated system is autonomous, of order 1, and of degree 1 in the derivatives of the parameters  $s$  and  $t$ . Every non-trivial rational solution  $\mathcal{R}(x) = (s(x), t(x))$  of the associated system implicitly defines a curve  $G(s, t)$  s.t.  $G(s(x), t(x)) = 0$ . By differentiation and taking into account (5), we get that the polynomial  $G_s \cdot f_1 + G_t \cdot f_2$  vanishes at  $\mathcal{R}(x)$ ; so  $G_s \cdot f_1 + G_t \cdot f_2 \in \langle G \rangle$ . Such curves  $G(s, t)$  are called *invariant algebraic curves*. The irreducible factors of an invariant curve are also invariant curves. A more detailed analysis can be found in [31]. In the generic case, in which the associated system has no dicritical points, Carnicer [5] gives an upper bound for irreducible invariant algebraic curves derived from the degree of the associated system (5). A rational parametrization of an invariant algebraic curve presents a candidate for a rational solution of the associated system, and therefore of the original AODE. A necessary and sufficient condition for a rational invariant algebraic curve to be a rational solution curve is given in [31], Theorem 3.5. Suppose  $s(x) \neq 0$  (otherwise proceed analogously with  $t(x)$ ; not both  $s$  and  $t$  can be constant). Then  $(s(T(x)), t(T(x)))$  is the rational solution of (5) corresponding to  $G$ , where the reparametrization  $T$  is the solution of

$$T' = \frac{1}{s'(T)} \cdot \frac{f_1(s(T), t(T))}{g(s(T), t(T))}.$$

This transformation  $T$  is always a Möbius transformation.

**Example 1 (continued):** We are considering the differential equation

$$F(x, y, y') \equiv y'^2 + 3y' - 2y - 3x = 0.$$



The solution surface  $z^2 + 3z - 2y - 3x = 0$  has the parametrization

$$\mathcal{P}(s, t) = \left( \frac{t}{s} + \frac{2s + t^2}{s^2}, -\frac{1}{s} - \frac{2s + t^2}{s^2}, \frac{t}{s} \right).$$

This is a proper parametrization and its associated system is

$$s' = st, \quad t' = s + t^2.$$

So  $f_1 = st, f_2 = s + t^2$ . An irreducible invariant algebraic curve of this system is, for example,  $G(s, t) = s^2 + ct^2 + 2cs$ . Indeed,  $G_s f_1 + G_t f_2 = 2t \cdot G$ . Now,  $G(s, t) = s^2 + ct^2 + 2cs = 0$  depends on a transcendental parameter  $c$ , and it can be parametrized by

$$\mathcal{Q}(x) = \left( -\frac{2c}{1 + cx^2}, -\frac{2cx}{1 + cx^2} \right).$$

The differential equation defining the reparametrization is  $T' = 1$ . Hence  $T(x) = x$ . So the rational solution in this case is

$$s(x) = -\frac{2c}{1 + cx^2}, \quad t(x) = -\frac{2cx}{1 + cx^2}.$$

Since  $G(s, t)$  contains a transcendental constant, the above solution is a rational general solution of the associated system. Therefore, the rational general solution of  $F(x, y, y') = 0$  is

$$y = \frac{1}{2}x^2 + \frac{1}{c}x + \frac{1}{2c^2} + \frac{3}{2c},$$

which, after a change of parameter, can be written as

$$y = \frac{1}{2}((x + c)^2 + 3c). \quad \square$$

In most cases this approach can compute a general rational solution if it exists; but the dicritical case presents a problem. So Approach 1 is not a decision algorithm.

Approach 2: Alternatively, we could associate with the AODE  $F(x, y, y') = 0$  an algebraic curve in  $\mathbb{A}^2(\overline{\mathbb{Q}}(x))$  by taking the coefficient field to be  $\mathbb{Q}(x)$ ; then again we relate rational solutions of the AODE with rational parametrizations of the curve. Approach 2 is developed in [41].

For a given first-order AODE  $F(x, y, y') = 0$  over  $K$ , we consider the *corresponding algebraic curve*  $\mathcal{C}_F$  defined by the algebraic equation  $F(x, y, z) = 0$  over  $K(x)$ . I.e., we consider the algebraic equation  $F(y, z) = 0$ , with  $F(y, z) \in K(x)[y, z]$ .

We change the problem slightly. The arbitrary constant  $c$  might appear algebraically in a general rational solution. But now we want rational general

solutions  $y(x)$ , in which the arbitrary constant  $c$  appears rationally. Such a solution is called a *strong rational general solution*. The rational general solutions computed in Examples 1 and 2 are in fact strong rational general solutions. But the AODE

$$x^3 y'^3 - (3x^2 y - 1)y'^2 + 3xy^2 y' - y^3 + 1 = 0$$

has the rational general solution  $y(x) = cx + (c^2 + 1)^{\frac{1}{3}}$ , which is not strong. The curve  $\mathcal{C}_F$  has genus 1. So this AODE does not have a strong rational general solution.

The existence of a strong rational general solution implies the existence of a rational parametrization of the algebraic curve  $\mathcal{C}_F$  with coefficients in  $K(x)$ ; i.e., without algebraically extending the coefficient field  $K(x)$  to  $\overline{K(x)}$ . We call such a parametrization an optimal parametrization. In [41] we prove that if the AODE  $F(x, y, y') = 0$  has a strong rational solution, then  $\mathcal{C}_F$  is rational and it can be parametrized without any algebraic field extension.

Given an optimal proper parametrization  $\mathcal{P}(t) = (p_1(t), p_2(t))$  of the curve  $\mathcal{C}_F$ , we can turn the AODE  $F(x, y, y') = 0$  into a quasi-linear *associated* differential equation of the form

$$\omega' = a_0(x) + a_1(x)\omega + a_2(x)\omega^2, \quad \text{with } a_i \in K(x),$$

i.e., a Riccati equation. The proof of this theorem depends on a result by Fuchs [11]: *If a quasi-linear ODE  $y' = f(x, y)$ , where  $f \in K(x, y)$ , has a rational general solution, then it must be a Riccati equation.* In case the parametrization is not optimal, it contains an algebraic function in  $x$ . This algebraic function then also appears in the associated quasi-linear ODE, and Fuchs' theorem would not be applicable. There is a 1-1 correspondence between rational general solutions of the original AODE and rational general solutions of the associated Riccati equation; in particular, if  $\omega(x)$  is a rational general solution of the associated equation, the  $y(x) = p_1(x, \omega(x))$  is a rational general solution of the original AODE. Schwarz [35] shows that if a Riccati equation has 3 special rational solutions, then it has a strong rational general solution. All these considerations finally lead to the result (Cor. 5.5 in [41]): *If a parametrizable first-order AODE has a rational general solution, then it has a strong rational general solution.*

In case  $a_2(x) = 0$ , the associated differential equation is linear, and it can be solved by integration. In case  $a_2(x) \neq 0$ , it is a classical Riccati equation and it can be solved by Kovacic's algorithm described in [22].

**Example 3:** (Example 1.537 in the collection of Kamke [20]) Consider the AODE

$$F(x, y, y') = (xy' - y)^3 + x^6 y' - 2x^5 y = 0.$$

The associated curve  $\mathcal{C}_F$  defined by  $F(x, y, z) = 0$  can be parametrized as

$$\mathcal{P}(t) = \left( -\frac{t^3x^5 - t^2x^6 + (t-x)^3}{t^3x^5}, -\frac{2t^3x^5 - 2t^2x^6 + (t-x)^3}{t^3x^6} \right).$$

Therefore, the associated differential equation w.r.t.  $\mathcal{P}$  is

$$\omega' = \frac{1}{x^2}\omega(2\omega - x),$$

which is a Riccati equation. Kovacic's algorithm gives us the rational general solution  $\omega(x) = \frac{x}{1+cx^2}$ . Hence, the original AODE  $F(x, y, y') = 0$  has the strong rational general solution

$$y(x) = cx(x + c^2).$$

□

Approach 2 provides a full decision algorithm for the existence of a strong rational general solution of a parametrizable AODE; 64 percent of all the first-order ODEs in the collection of Kamke fall into this class. In the positive case, we can actually determine a strong rational general solution.

The algebro-geometric approach for AODEs has also been applied to other problems, for which we might want to decide the existence of rational solution; examples are AODEs of higher order [18], or 1-dimensional systems of AODEs [23]. Recently we have investigated the problem of determining all rational solutions of an AODE of order 1; cf. [42]. For two classes of AODEs (parametrizable and maximally comparable) we actually give algorithms for computing all rational solutions. Note that in the collection of Kamke [20] there are only 3 AODEs of order 1 (1.372, 1.545, 1548) which are neither parametrizable, nor maximally comparable. These are all autonomous and non-parametrizable, and hence cannot have a rational solution.

### 3 Algebraic solutions of AODEs

Aroca et. al [1] presented a polynomial time algorithm for computing algebraic general solutions of autonomous AODEs  $F(y, y') = 0$ . Vo and Winkler (in [43]) modified Approach 1 of Section 2 in order to compute algebraic general solutions of non-autonomous parametrizable AODEs  $F(x, y, y') = 0$ . But lacking a degree bound for such algebraic solutions, they need to specify a bound for the algebraic extension degree.

As in Section 2 one reduces the problem of solving (2), for  $n = 1$ , to an associated planar rational system of the form

$$s' = M(s, t), \quad t' = N(s, t), \tag{7}$$

where  $M$  and  $N$  are rational functions in  $s$  and  $t$ . If the parametrizable AODE  $F(x, y, y') = 0$  has an algebraic general solution, then its associated system w.r.t. a proper parametrization has a *rational first integral*; i.e. a rational function  $W(s, t)$  s.t.  $M \frac{\partial W}{\partial s} + N \frac{\partial W}{\partial t} = 0$ ; cf. Proposition 3 in [43]. Furthermore, by Corollary 1 in [43], if  $W = \frac{P}{Q}$  is a reduced rational first integral of the associated system and  $(s(x), t(x))$  is an algebraic general solution, then  $s(x)$  is an algebraic general solution of the autonomous first order AODE  $F_1(s, s') = 0$ , where  $F_1(s, r) := \text{res}_t(P(s, t) - cQ(s, t), rM_2(s, t) - M_1(s, t))$ ; analogously for  $t(x)$ .

We still need to bound the degree of the algebraic solution. This is achieved in Theorem 5 of [43], which says that if  $F(x, y, y') = 0$  has an algebraic solution with minimal polynomial of degree less or equal to  $n$ , then the associated system has a rational first integral of degree  $m(n)$ ; an explicit formula for  $m(n)$  is given. So we can decide the existence of an algebraic solution having extension degree less or equal to  $n$ .

**Example 4:** (from [43]) Consider the differential equation

$$4x(x - y)y'^2 + 2xyy' - 5x^2 + 4xy - y^2 = 0. \quad (8)$$

The solution surface of the differential equation is rational. It is parametrized by the rational map

$$\mathcal{P}(s, t) := \left( s, -\frac{t^2 - 5ts + 5s^2}{s}, \frac{t^2 - 4st + 5s^2}{2s(t - 2s)} \right)$$

The associated system with respect to  $\mathcal{P}$  is

$$s' = 1, \quad t' = \frac{t^2 - 3s^2}{2s(t - 2s)}.$$

If we look for an algebraic general solution  $y(x)$  of degree at most  $n = 2$ , we need to find a rational first integral of degree at most  $m(n) = 32$  of the associated system. In this case, the associated system has the rational first integral  $W = \frac{t^2 - 4st + 3s^2}{s}$ , which is of total degree 2. (This suggests that our degree bound is not optimal). Thus it has an algebraic solution  $(s(x), t(x)) := (x, \bar{t}(x, c))$ , where  $\bar{t}(x, c)$  is a root of the algebraic equation  $t^2 - 4xt + 3x^2 - cx = 0$ . By applying (6), we see that

$$y(x) = \frac{\sqrt{cx(cx + 1)} - 1}{c}$$

is an algebraic general solution of the differential equation.  $\square$

#### 4 Classification of AODEs w.r.t. transformation groups

Consider a group of transformations leaving the associated system of an AODE invariant. We call such a transformation *solution preserving*. Orbits w.r.t. such a

transformation group contain AODEs of equal complexity in terms of determining rational solutions. In [28, 29] we have studied affine and birational transformations and investigated their usefulness within the algebro-geometric solution method. For instance, it turns out that being autonomous is not an intrinsic property of an AODE; certain classes contain both autonomous and non-autonomous AODEs.

The group  $\mathcal{G}_a$  of affine transformations

$$L : \mathbb{A}^3(\mathbb{K}) \longrightarrow \mathbb{A}^3(\mathbb{K})$$

$$v \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ b & a & 0 \\ 0 & 0 & a \end{pmatrix} v + \begin{pmatrix} 0 \\ c \\ b \end{pmatrix}$$

leaves the associated system of an AODE invariant, and therefore also the rational solvability. The group  $\mathcal{G}_a$  defines a group action on AODEs by

$$\mathcal{G}_a \times \mathcal{AODE} \rightarrow \mathcal{AODE}$$

$$(L, F) \mapsto L \cdot F = (F \circ L^{-1})(x, y, y') .$$

Let  $F$  be a parametrizable AODE, and  $L \in \mathcal{G}_a$ . For every proper rational parametrization  $\mathcal{P}$  of the solution surface  $F(x, y, z) = 0$ , the associated system of  $F(x, y, y') = 0$  w.r.t.  $\mathcal{P}$  and the associated system of  $(L \cdot F)(x, y, y') = 0$  w.r.t.  $L \circ \mathcal{P}$  are equal.

**Example 5:** We consider the AODE from Example 1

$$F(x, y, y') \equiv y'^2 + 3y' - 2y - 3x = 0 .$$

We first check whether in the orbit of  $F$  there exists an autonomous AODE. For this, we apply a generic  $L$  to  $F$  to get

$$(L \cdot F)(x, y, y') = \frac{1}{a^2}y'^2 + \frac{3}{a}y' - \frac{2b}{a^2}y' - \frac{2}{a}y + \frac{2b}{a}x - 3x - \frac{3b}{a} + \frac{b^2}{a^2} + \frac{2c}{a} .$$

Therefore, for every  $a \neq 0$  and  $b$  such that  $2b - 3a = 0$ , we get an autonomous AODE. In particular, for  $a = 1$ ,  $b = 3/2$ , and  $c = 0$  we get

$$L = \left[ \begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{3}{2} \end{pmatrix} \right] ,$$

i.e., we obtain

$$F(L^{-1}(x, y, y')) \equiv y'^2 - 2y - \frac{9}{4} = 0 .$$

So by the algorithm of Feng and Gao we may solve this transformed autonomous AODE and then transform the solution back to the original equation.  $\square$

The group  $\mathcal{G}_b$  of birational transformations from  $\mathbb{K}^3$  to  $\mathbb{K}^3$  of the form

$$\Phi(u_1, u_2, u_3) = \left( u_1, \frac{au_2 + b}{cu_2 + d}, \frac{\partial}{\partial u_1} \left( \frac{au_2 + b}{cu_2 + d} \right) + \frac{\partial}{\partial u_2} \left( \frac{au_2 + b}{cu_2 + d} \right) \cdot u_3 \right) ,$$

where  $a, b, c, d \in \mathbb{K}[u_1]$  such that  $ad - bc \neq 0$ , defines a group action on  $\mathcal{AODE}$ ; we let  $\Phi \cdot F$  be the primitive part w.r.t.  $y$  and  $y'$  of the numerator of  $(F \circ \Phi^{-1})(x, y, y')$ ; see Theorem 3.3 in [29]. These birational transformations leave the associated system of an AODE invariant, and therefore also the rational solvability.

**Example 6:** Consider the AODE

$$F(x, y, y') = 25x^2y'^2 - 50xyy' + 25y^2 + 12y^4 - 76xy^3 + 168x^2y^2 - 144x^3y + 32x^4 = 0 .$$

Using the transformation

$$\Phi(u, v, w) = \left( u, \frac{u - 3v}{-2u + v}, \frac{-5v}{(2u - v)^2} + \frac{5u}{(2u - v)^2} w \right)$$

we get the autonomous equation

$$G(y, y') = F(\Phi^{-1}(x, y, y')) = y'^2 - 4y = 0 .$$

Observe that  $F$  cannot be transformed into an autonomous AODE by affine transformations. The rational general solution  $y = (x + c)^2$  of  $G(y, y') = 0$  is transformed into

$$y = \frac{x(2(x + c)^2 + 1)}{(x + c)^2 + 3} ,$$

the rational general solution of  $F(x, y, y') = 0$ .  $\square$

So-called strict equivalence for AODEs of order 1 with coefficients in a finite extension  $K$  of the rational functions over  $\mathbb{C}$  is considered in [27].

## 5 Extending the reach of the algebro-geometric method

In [15] we also consider other types of solutions of autonomous AODEs of order 1. In particular, we investigate radical solutions. This class of solutions is more general than rational solutions, but more special than algebraic solutions. We generalize the criterion of Feng and Gao to this situation; i.e., if  $\mathcal{P}(t) = (r(t), s(t))$  is a radical parametrization of the curve  $F(y, z) = 0$  and  $s(t)/r'(t)$  has a certain

shape, then we can determine a radical general solution of the AODE  $F(y, y') = 0$ . But this approach does not lead to a decision algorithm for the existence of a radical solution. Also, in [15] examples are given in which the algebro-geometric method leads to transcendental general solutions.

AODEs of order higher than 1 are considered in [17, 18]. Many of the ideas in the algebro-geometric method can be generalized to this situation. But one of the main new problems is that for hypersurfaces of dimension 3 or more, rationality does not necessarily imply birationality; cf. [2]. So we loose the important property that a rational solution is a proper parametrization and that all proper parametrizations are related by a well-defined group action, such as Möbius transformations.

First steps have been taken towards extending the algebro-geometric solution method to algebraic partial differential equations. We consider autonomous APDEs in [12], and systems of non-autonomous APDEs in [13]. But these first results definitely need further analysis, and a decision algorithm or even an algorithmic treatment is not in reach.

## 6 Conclusion

We have seen that the algebro-geometric method for solving algebraic differential equations is a fruitful approach to actually creating formula solutions for certain kinds of differential equations. Besides giving a decision method for rational or algebraic autonomous AODEs of order 1, it also lets us decide whether a parametrizable non-autonomous AODE of order 1 has a general rational solution, which in this case has to be a strong rational general solution. However, we still do not have a full algorithm for deciding the existence of a rational or algebraic general solution and in the positive case computing it for an arbitrary AODE of order 1. For a big class of AODEs of order 1 we can compute all rational solutions. First steps have been taken towards the treatment of AODEs of higher order, and we have started to create a theoretical framework for discussing how to determine rational solutions of APDEs.

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