Chapter 4
Differential Resultants

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2. Elementary background material – Sylvester resultant

Let R be a commutative ring with identity element 1. In the first subsection we shall review the definition of the classical Sylvester resultant $\mathrm{res}(f,g)$ of $f(x), g(x) \in R[x]$. We shall state the requirements on R so that $\mathrm{res}(f,g) = 0$ is a necessary and sufficient condition for the existence in some extension of R of a solution α to the system

$$f(x) = g(x) = 0 \ .$$

In the second subsection we review elementary differential algebra on R. In particular we define the notion of a derivation on R, and introduce the ring of differential polynomials over R. The elementary background concepts from this section will provide the foundation for the theory of the differential Sylvester resultant, developed in the next section.

Sylvester resultant

We assume at the outset that R is an integral domain (commutative ring with 1, and no zero divisors), and that K is its quotient field.

Let

$$f(x) = \sum_{i=0}^{m} a_i x^i, \quad g(x) = \sum_{j=0}^{n} b_j x^j$$

be polynomials of positive degrees m and n, respectively, in $R[x]$.

If f and g have a common factor $d(x)$ of positive degree, then they have a common root in the algebraic closure $\overline{K}$ of K; so the system of equations

$$f(x) = g(x) = 0 \tag{1}$$

has a solution in $\overline{K}$.

On the other hand, if $\alpha \in \overline{K}$ is a common root of f and g, then $\mathrm{norm}_{K(\alpha):K}(x - \alpha)$ is a common divisor of f and g in $K[x]$. So, by Gauss' Lemma (for which we need R to be a unique factorization domain) on primitive polynomials there is a similar (only differing by a factor in K) common factor of f and g in $R[x]$.

We summarize these observations as follows:

Proposition 2.1. *Let R be a unique factorization domain (UFD) with quotient field K. For polynomials $f(x), g(x) \in R[x]$ the following are equivalent:*

(i) *f and g have a common solution in $\overline{K}$, the algebraic closure of K,*

(ii) *f and g have a common factor of positive degree in $R[x]$.*

So now we want to determine a necessary condition for f and g to have a common divisor of positive degree in $R[x]$. Suppose that f and g indeed have a common divisor $d(x)$ of positive degree in $R[x]$; i.e.,

$$f(x) = d(x)\overline{f}(x), \quad g(x) = d(x)\overline{g}(x). \tag{2}$$

Then for $p(x) := \overline{g}(x)$, $q(x) := -\overline{f}(x)$ we have

$$p(x)f(x) + q(x)g(x) = 0. \tag{3}$$

So there are non-zero polynomials p and q with $\deg p < \deg g, \deg q < \deg f$, satisfying equation (3).

This means that the linear system

$$\begin{pmatrix} p_{n-1} & \cdots & p_0 & q_{m-1} & \cdots & q_0 \end{pmatrix} \cdot \begin{pmatrix} A \\ \cdots \\ B \end{pmatrix} = 0 \,, \qquad (4)$$

where

$$A = \begin{pmatrix} a_m & a_{m-1} & \cdots & a_0 & & & & \\ & a_m & a_{m-1} & \cdots & a_0 & & & \\ & & \ddots & & & & \ddots & \\ & & & & a_m & a_{m-1} & \cdots & a_0 \end{pmatrix} \in R^n_{m+n} \quad ,$$

$$B = \begin{pmatrix} b_n & b_{n-1} & \cdots & b_0 & & & & \\ & b_n & b_{n-1} & \cdots & b_0 & & & \\ & & \ddots & & & & \ddots & \\ & & & & b_n & b_{n-1} & \cdots & b_0 \end{pmatrix} \in R^m_{m+n} \,,$$

has a non-trivial solution.

The matrix of this system (4) is called the *Sylvester matrix* of f and g. Thus, the determinant of the Sylvester matrix of f and g is 0. The *resultant* of f and g, $\mathrm{res}(f, g)$, is this determinant, and it is clear that the resultant is a polynomial expression of the coefficients of f and g, and therefore an element of the integral domain R. This does not require R to be a UFD. Summarizing:

Proposition 2.2. *Let $f, g \in R[x]$, for R an integral domain. $\mathrm{res}(f, g) = 0$ is a* **necessary condition** *for f and g to have a common factor of positive degree.*

If we identify a polynomial of degree d with the vector of its coefficients of length $d+1$, we may also express this in terms of the linear map

$$\begin{array}{rlcl} S: & K^{m+n} & \longrightarrow & K^{m+n} \\ & (p_{n-1}, \ldots, p_0, q_{m-1}, \ldots, q_0) & \mapsto & \text{coefficients of } pf + qg \end{array}$$

Obviously the existence of a non-trivial linear combination (3) is equivalent to S having a non-trivial kernel, and therefore to S having determinant 0.

But is the vanishing of the resultant also a sufficient condition for f and g to have a common factor of positive degree? Suppose that $\mathrm{res}(f, g) = 0$. This means that (3) has a non-trivial solution $u(x), v(x)$ of bounded degree; so

$$u(x)f(x) = -v(x)g(x) \ .$$

These co-factors will be in $K[x]$, but of course we can clear denominators and have a similar relation with co-factors in $R[x]$. If we now require the coefficient domain R to be a unique factorization domain (UFD), we see that every irreducible factor of f must appear on the right hand side with at least the same multiplicity. Not all of these factors can be subsumed in v, because v is of lower degree than f. So at least one of the irreducible factors of f must divide g. Thus we have:

Proposition 2.3. *Let $f, g \in R[x]$, for R a UFD. $\mathrm{res}(f, g) = 0$ is also a* **sufficient condition** *for f and g to have a common factor of positive degree; and therefore a common solution in $\overline{K}$.*

A further property is of interest and importance.

Proposition 2.4. *The resultant is a constant in the ideal generated by f and g in $R[x]$; i.e. we can write*

$$\mathrm{res}(f, g) \ = \ u(x)f(x) + v(x)g(x), \tag{5}$$

with $u, v \in R[x]$. Moreover, these cofactors satisfy the degree bounds $\deg(u) < \deg(g)$, $\deg(v) < \deg(f)$.

Proof: We follow an argument given in [4]. In fact, Collins proves this fact for R being a general commutative ring with 1.

Consider the Sylvester matrix $S = (A \vdots B)^T$; i.e. the $(m+n) \times (m+n)$ matrix, whose first n rows consist of the coefficients of

$$x^{n-1} \cdot f(x), \ \dots \ , x \cdot f(x), \ f(x) \ ,$$

and whose last m rows consist of the coefficients of

$$x^{m-1} \cdot g(x), \ \dots \ , x \cdot g(x), \ g(x) \ .$$

Now, for $1 \leq i < m+n$, multiply the ith column of S by x^{m+n-i} and add to the last column. This will result in a new matrix T, having the same determinant as S. The columns of T are the same as the corresponding columns of S, except for the last column, which consists of the polynomials

$$x^{n-1} \cdot f(x), \ \dots \ , x \cdot f(x), \ f(x), \ x^{m-1} \cdot g(x), \ \dots \ , x \cdot g(x), \ g(x) \ .$$

Expanding the determinant of T w.r.t. its last column, we obtain polynomials $u(x)$ and $v(x)$ satisfying the relation (5), and also the degree bounds. $\square$

An alternative approach (similar to that above but with a slightly different emphasis) to defining the Sylvester resultant of $f(x)$ and $g(x)$ is to regard all the coefficients a_i and b_j of f and g as distinct and unrelated indeterminates. The indeterminates a_m and b_n are then referred to as the *formal* leading coefficients of f and g, respectively. In effect we take R to be the domain $\mathbb{Z}[a_m, \ldots, a_0, b_n, \ldots, b_0]$. This approach allows us to study the resultant $\operatorname{res}(f, g)$ as a polynomial in the $m + n + 2$ indeterminates a_i and b_j. Indeed it is not hard to see that $\operatorname{res}(f, g)$ is homogeneous in the a_i of degree n, homogeneous in the b_j of degree m, and has the "principal term" $a_m^n b_0^m$ (from the principal diagonal). With this approach, adjustment of some of the basic facts is needed. For example, the analogue of Proposition 2.3 would state that, for D a UFD, after replacement of all the coefficients a_i and b_j by elements of D, $\operatorname{res}(f, g) = 0$ is a sufficient condition for either $f(x)$ and $g(x)$ to have a common factor of positive degree, or $a_m = b_n = 0$.

Another variation on defining the Sylvester resultant of two polynomials is to start instead with two *homogeneous* polynomials $F(x,y) = \sum_{i=0}^{m} a_i x^i y^{m-i}$ and $G(x,y) = \sum_{j=0} b_j x^j y^{n-j}$. Let us similarly regard the coefficients a_i and b_j as indeterminates. Then the resultant of F and G is defined as $\mathrm{res}(F,G) = \mathrm{res}(f,g)$, where $f(x) = F(x,1)$ and $g(x) = G(x,1)$. Our analogue of Proposition 2.3 then becomes simpler. Combining it with homogeneous analogues of Propositions 2.1 and 2.2 we have:

Proposition 2.5. *After assigning values to the coefficients from a UFD D, $\mathrm{res}(F,G) = 0$ is a necessary and sufficient condition for $F(x,y)$ and $G(x,y)$ to have a common factor of positive degree over D, hence for a common zero to exist over an extension of the quotient field of D.*

An ideal I of a differential ring R is known as a *differential ideal* if $r \in I$ implies $r' \in I$. If $r_1, \ldots, r_n \in R$ we denote by $\langle r_1, \ldots, r_n \rangle$ the differential ideal generated by $r_1, \ldots, r_n$, that is, the ideal generated by the r_i and all their derivatives.

Example 2.6. The familiar rings such as $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are differential rings if we set $\partial(a) = 0$ for all elements a.

Example 2.7. Let K be a field and t an indeterminate over K. Then $K[t]$, equipped with the derivation $\partial = d/dt$, is a differential integral domain and its quotient field $K(t)$ is a differential field, again with standard differentiation as its derivation. K is the ring (field) of constants of $K[t]$ $(K(t))$.

Example 2.8. Let (R, ∂) be a differential ring. Let $x = x^{(0)}, x^{(1)}, x^{(2)}, \dots$ be distinct indeterminates over R. Put $\partial(x^{(i)}) = x^{(i+1)}$ for all $i \geq 0$. Then ∂ can be extended to a derivation on the polynomial ring $R\{x\} := R[x^{(0)}, x^{(1)}, \dots]$ in a natural way, and we denote this extension also by ∂. The ring $R\{x\}$ together with this extended ∂ is a differential ring, called the ring of *differential polynomials* in the differential indeterminate x over R. An element $f(x) = \sum_{i=0}^{m} a_i x^{(i)}$ of $R\{x\}$ with $a_m \neq 0$ has *order* m and *leading coefficient* a_m.

(**Remark**. It may be helpful to think of elements of R and of $x, x^{(1)}, \dots$ as functions of an indeterminate t, and to regard ∂ as differentiation with respect to t.) If (K, ∂) is a differential field then $K\{x\}$ is a differential integral domain, and its derivation extends uniquely to the quotient field. We write $K\langle x\rangle$ for this quotient field; its elements are *differential rational functions* of x over K.

3. Differential Sylvester resultant

Let (R, ∂) be a differential integral domain. Recall from Section 2 that the ring (indeed domain) of differential polynomials in the differential indeterminate x is denoted by $R\{x\}$. Then $R\{x\}$ is also a (left) R-module, and we denote by $R_{LH}\{x\}$ the R-submodule comprising those elements of $R\{x\}$ which are linear and homogeneous. We aim in this section to define a certain resultant, known as a differential Sylvester resultant, of two elements of $R_{LH}\{x\}$. We shall begin by studying a closely related noncommutative ring: namely, we consider the ring $R[\partial]$ of linear differential operators on R. As we shall see, there is an important relationship between $R[\partial]$, considered as left R-module, and $R_{LH}\{x\}$: these are isomorphic as left R-modules. Thus the differential theory of $R[\partial]$ and $R_{LH}\{x\}$ can to an extent be developed in parallel. The details are provided in the next two subsections.

We consider the ring of linear differential operators $R[\partial]$, where the application of $A = \sum_{i=0}^{m} a_i \partial^i$ to $r \in R$ is defined as

$$A(r) = \sum_{i=0}^{m} a_i r^{(i)} \ .$$

Here $r^{(i)}$ denotes the i-fold application of ∂ (that is, $'$) to r. If $a_m \neq 0$, the *order* of A is m and a_m is the *leading coefficient* of A. Now the application of A can naturally be extended to K, and to any extension of K. If $A(\eta) = 0$, with η in R, K or any extension of K, we call η a *root* of the linear differential operator A.

The application of the constant operator r to a yields $r(a) = r \cdot a^{(0)} = r \cdot a$.

The ring $R[\partial]$ is non-commutative; let us see what the commutation rule is. If we apply ∂r to a we get

$$\partial r(a) = \partial(ra) = r\partial(a) + (\partial(r))a = r\partial(a) + r'a = (r\partial + r')(a).$$

So the corresponding rule for the multiplication of ∂ by an element of $r \in R$ is

$$\partial r = r\partial + r' \ .$$

Note that ∂r, which denotes the operator product of ∂ and r, is distinct from $\partial(r)$ (that is, from r'), the application of map ∂ to r.

Proposition 3.1. *For $n \in N$:* $\partial^n r \ = \ \sum_{i=0}^{n} \binom{n}{i} r^{(n-i)} \partial^i$.

Proof: For $n = 0$ this obviously holds.
Assume the fact holds for some $n \in \mathbb{N}$. Then

$$
\begin{array}{rcl}
\partial^{n+1} r & = & \partial(\partial^n r) \ = \ \partial\big(\sum_{i=0}^{n} \binom{n}{i} r^{(n-i)} \partial^i\big) \\
 & = & \sum_{i=0}^{n} \binom{n}{i} \partial r^{(n-i)} \partial^i \ = \ \sum_{i=0}^{n} \binom{n}{i} [r^{(n-i)} \partial + r^{(n-i+1)}] \partial^i \\
 & = & \sum_{i=0}^{n} \binom{n}{i} r^{(n-i)} \partial^{i+1} \ + \ \sum_{i=0}^{n} \binom{n}{i} r^{(n-i+1)} \partial^i \\
 & = & \sum_{i=1}^{n+1} \binom{n}{i-1} r^{(n+1-i)} \partial^i \ + \ \sum_{i=0}^{n} \binom{n}{i} r^{(n-i+1)} \partial^i \\
 & = & \binom{n}{n} r^{(0)} \partial^{n+1} \ + \ \sum_{i=1}^{n} [\binom{n}{i-1} + \binom{n}{i}] r^{(n+1-i)} \partial^i \ + \ \binom{n}{0} r^{(n+1)} \partial^0 \\
 & = & \sum_{i=0}^{n+1} \binom{n+1}{i} r^{(n+1-i)} \partial^i. \qquad \square
\end{array}
$$

From a linear homogeneous ODE $p(x) = 0$, with $p(x) \in R\{x\}$,

$$p(x) = p_0(t)x + p_1(t)x' + \cdots + p_n(t)x^{(n)} = 0 \ ,$$

we can extract a linear differential operator

$$\mathcal{O}(p) = A = \sum_{i=0}^{n} p_i \partial^i \ ,$$

such that the given ODE can be written as

$$A(x) = 0,$$

in which x is regarded as an unknown element of R, K or some extension of K. Such a linear homogeneous ODE always has the trivial solution $x = 0$; so a linear differential operator always has the trivial root 0.

In [3] it is stated that $K[\partial]$ is left-Euclidean, and a few brief remarks are provided by way of proof. It follows from the left-Euclidean property that every left-ideal $_K I$ of the form $_K I = (A, B)$ is principle, and is generated by the right-gcd of A and B. As remarked in [3] with reference to [5], under suitable assumptions on K, any linear differential operator of positive order has a root in some extension of K. We state this result precisely.

Theorem 3.2. (Ritt-Kolchin). *Assume that the differential field K has characteristic 0 and that its field C of constants is algebraically closed. Then, for any linear differential operator A over K of positive order n, there exist n roots $\eta_1, \ldots, \eta_n$ in a suitable extension of K, such that the η_i are linearly independent over C. Moreover, the field $K\langle\eta_1, \ldots, \eta_n\rangle$ contains no constant not in C.*

This result is stated and proved in [7] using results from [6] and [8]. The field $K\langle\eta_1, \ldots, \eta_n\rangle$ associated with A is known as a *Picard-Vessiot extension* of K (for A).
Henceforth assume the hypotheses of Theorem 3.2.

It follows from Theorem 3.2 (Ritt-Kolchin) that if the operators $A, B \in K[\partial]$ have a common factor F of positive order on the right, i.e.,

$$A = \overline{A} \cdot F, \quad \text{and } B = \overline{B} \cdot F, \tag{6}$$

then they have a non-trivial common root in a suitable extension of K. For by Theorem 3.2, F has a root $\eta \neq 0$ in an extension of K. We have $A(\eta) = \bar{A}(F(\eta)) = \bar{A}(0) = 0$ and similarly $B(\eta) = 0$.

On the other hand, if A and B have a non-trivial common root η in a suitable extension of K, we show that they have a common right factor of positive order in $K[\partial]$. Let F be a nonzero differential operator of lowest order s.t. $F(\eta) = 0$. Then F has positive order. Because the ring of operators is left-Euclidean, F is unique up to multiplication of non-zero elements of K. This F is a right divisor of both A and B. To see this, apply division in the left-Euclidean ring $K[\partial]$:

$$A = Q \cdot F + R,$$

with the order of R less than the order of F, or $R = 0$. Apply both sides of this equation to η:

$$A(\eta) = (Q \cdot F)(\eta) + R(\eta).$$

Since $A(\eta) = 0$ and $F(\eta) = 0$, $R(\eta) = 0$. Therefore, by minimality of F, $R = 0$. Hence F is a right divisor of A. We see that F is a right divisor of B similarly.

We summarize our result in the following theorem, which is the closest analogue of Proposition 2.1 we can state.

Theorem 3.3. *Assume that K has characteristic 0 and that its field of constants is algebraically closed. Let A, B be differential operators of positive orders in $K[\partial]$. Then the following are equivalent:*

(i) *A and B have a common non-trivial root in an extension of K,*

(ii) *A and B have a common factor of positive order on the right in $K[\partial]$.*

Now let us see that the existence of a non-trivial factor (6) is equivalent to the existence of a non-trivial order-bounded linear combination

$$CA + DB = 0 \ , \tag{7}$$

with $\mathrm{order}(C) < \mathrm{order}(B)$ and $\mathrm{order}(D) < \mathrm{order}(A)$, and $(C, D) \neq (0, 0)$.
For given $A, B \in K[\partial]$, with $m = \mathrm{order}(A), n = \mathrm{order}(B)$, consider the linear map

$$\begin{array}{rlcl} S: & K^{m+n} & \longrightarrow & K^{m+n} \\ & (c_{n-1}, \ldots, c_0, d_{m-1}, \ldots, d_0) & \mapsto & \text{coefficients of } CA + DB \end{array} \tag{8}$$

Obviously the existence of a non-trivial linear combination (7) is equivalent to S having a non-trivial kernel, and therefore to S having determinant 0. Indeed we have the following result.

Theorem 3.4. $\det(S) = 0$ *if and only if A and B have a common factor (on the right) in $K[\partial]$ of positive order.*

Proof: Suppose $\det(S) = 0$. This means that S cannot be surjective. Now the right-gcd G of A and B can be written as an order-bounded linear combination of A and B, so it is in the image of the map S. This means that G cannot be trivial (that is, G cannot be an element of K), because otherwise S would be surjective.

On the other hand, suppose that $\det(S) \neq 0$. Then the linear map is invertible; in particular, it is surjective. Therefore there exist $C, D \in K[\partial]$ with appropriate degree bounds, s.t. $1 = CA + DB$. So every common divisor (on the right) of A and B is a common divisor of 1. Therefore no common divisor of A and B could have positive order. $\square$

So let us see which linear conditions on the coefficients of A and B we get by requiring that (7) has a non-trivial solution of bounded order, i.e.,

$$\mathrm{order}(C) < \mathrm{order}(B) \quad \text{and} \quad \mathrm{order}(D) < \mathrm{order}(A).$$

Example 3.5. $\mathrm{order}(A) = 2 = \mathrm{order}(B)$

$$(c_1\partial + c_0)(a_2\partial^2 + a_1\partial + a_0) \;+\; (d_1\partial + d_0)(b_2\partial^2 + b_1\partial + b_0)$$

order 3:

$$c_1\partial a_2\partial^2 = c_1(a_2\partial + a_2')\partial^2 = c_1 a_2\partial^3 + c_1 a_2'\partial^2$$

$$d_1\partial b_2\partial^2 = d_1(b_2\partial + b_2')\partial^2 = d_1 b_2\partial^3 + d_1 b_2'\partial^2$$

order 2:

$$c_1 a_2'\partial^2 \text{ (from above)} + c_1\partial a_1\partial + c_0 a_2\partial^2 = c_1 a_2'\partial^2 + c_1 a_1\partial^2 + c_0 a_2\partial^2 + c_1 a_1'\partial$$

analogous for b and d

order 1:

$$c_1 a_1'\partial \text{ (from above)} + c_1\partial a_0 + c_0 a_1\partial = c_1 a_1'\partial + c_1(a_0\partial + a_0') + c_0 a_1\partial$$

$$= c_1 a_1'\partial + c_1 a_0\partial + c_0 a_1\partial + c_1 a_0'$$

analogous for b and d

order 0:

$$c_1 a_0' \text{ (from above)} + c_0 a_0$$

$$\text{analogous for } b \text{ and } d$$

So, finally,

$$\begin{pmatrix} c_1 & c_0 & d_1 & d_0 \end{pmatrix} \cdot \begin{pmatrix} a_2 & a_1 + a_2' & a_0 + a_1' & a_0' \\ 0 & a_2 & a_1 & a_0 \\ b_2 & b_1 + b_2' & b_0 + b_1' & b_0' \\ 0 & b_2 & b_1 & b_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} .$$

Observe, that the rows of this matrix consist of the coefficients of

$$\partial A, \ A, \ \partial B, \ B \ .$$

Comparing this to the example in [3], p.3, we see that after interchanging of rows this is the same matrix. $\square$

Example 3.6. $\mathrm{order}(A) = 2, \mathrm{order}(B) = 3$

$$(c_2\partial + c_1\partial + c_0)(a_2\partial^2 + a_1\partial + a_0) \;+\; (d_1\partial + d_0)(b_3\partial^3 + b_2\partial^2 + b_1\partial + b_0)$$

order 4:

$$c_2\partial^2 a_2\partial^2 + d_1\partial b_3\partial^3 = 0$$

$$\underline{a_2c_2\partial^4} + 2a_2'c_2\partial^3 + a_2''c_2\partial^2 + \underline{b_3d_1\partial^4} + b_3'd_1\partial^3 = 0$$

order 3:

$$(2a_2'c_2\partial^3 + a_2''c_2\partial^2 \text{ from above}) + c_2\partial^2 a_1\partial + c_1\partial a_2\partial^2 + (b_3'd_1\partial^3 \text{ from above}) + d_1\partial b_2\partial^2 + d_0b_3\partial^3 = 0$$

$$\underline{2a_2'c_2\partial^3} + \underline{a_1c_2\partial^3} + \underline{a_2c_1\partial^3} + a_2''c_2\partial^2 + 2a_1'c_2\partial^2 + a_1''c_2\partial + \underline{b_3'd_1\partial^3} + \underline{b_2d_1\partial^3} + \underline{b_3d_0\partial^3} + b_2'd_1\partial^2 = 0$$

order 2:

$$\begin{aligned}(a_2''c_2\partial^2 + 2a_1'c_2\partial^2 + a_1''c_2\partial + a_2'c_1\partial^2 \text{ from above}) + c_2\partial^2 a_0 + c_1\partial a_1\partial + c_0a_2\partial^2 & \\ +(b_2'd_1\partial^2 \text{ from above}) + d_1\partial b_1\partial + d_0b_2\partial^2 & = 0\end{aligned}$$

$$\begin{aligned}\underline{a_2''c_2\partial^2} + \underline{2a_1'c_2\partial^2} + a_1''c_2\partial + \underline{a_2'c_1\partial^2} + \underline{a_0c_2\partial^2} + 2a_0'c_2\partial + a_0''c_2 + \underline{a_1c_1\partial^2} + a_1'c_1\partial + \underline{a_2c_0\partial^2} & \\ +\underline{b_2'd_1\partial^2} + \underline{b_1d_1\partial^2} + b_1'd_1\partial + \underline{b_2d_0\partial^2} & = 0\end{aligned}$$

order 1:

$$(a_1'' c_2\partial + 2a_0' c_2\partial + a_0'' c_2 + a_1' c_1\partial \text{ from above}) + c_1\partial a_0 + c_0 a_1\partial + (b_1' d_1\partial \text{ from above}) + d_1\partial b_0 + d_0 b_1\partial \ = 0$$

$$\underline{a_1'' c_2\partial} + \underline{2a_0' c_2\partial} + \underline{a_1' c_1\partial} + \underline{a_0 c_1\partial} + a_0' c_1 + \underline{a_1 c_0\partial} + a_0'' c_2 + \underline{b_1' d_1\partial} + \underline{b_0 d_1\partial} + b_0' d_1 + \underline{b_1 d_0\partial} \ = 0$$

order 0:

$$(\underline{a_0' c_1} + \underline{a_0'' c_2} \text{ from above}) + \underline{a_0 c_0} + (\underline{b_0' d_1} \text{ from above}) + \underline{b_0 d_0} \ = 0.$$

So, finally

$$\begin{pmatrix} c_2 & c_1 & c_0 & d_1 & d_0 \end{pmatrix} \cdot \begin{pmatrix} a_2 & a_1 + 2a_2' & a_0 + 2a_1' + a_2'' & 2a_0' + a_1'' & a_0'' \\ 0 & a_2 & a_1 + a_2' & a_0 + a_1' & a_0' \\ 0 & 0 & a_2 & a_1 & a_0 \\ b_3 & b_2 + b_3' & b_1 + b_2' & b_0 + b_1' & b_0' \\ 0 & b_3 & b_2 & b_1 & b_0 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} .$$

Observe, that the rows of this matrix consist of the coefficients of

$$\partial^2 A, \ \partial A, \ A, \partial B, \ B \ . \qquad \square$$

Theorem 3.7. *The linear map S in* (8) *corresponding to* (7) *is given by the matrix whose rows are*
$\partial^{n-1}A, \dots, \partial A,\ A, \partial^{m-1}B, \dots, \partial B,\ B.$

Proof: Let $v = (c_{n-1}, \dots, c_0, d_{m-1}, \dots, d_0)$.
Consider an index i between 1 and n. If $c_{n-i} = 1$, and all the other components of v are 0, then v is mapped by S to $\partial^{n-i} \cdot A + 0 \cdot B = \partial^{n-i}A$. So the i-th row of S has to consist of the coefficients of $\partial^{n-i}A$.
Consider an index j between 1 and m. If $d_{m-j} = 1$, and all the other components of v are 0, then v is mapped by S to $0 \cdot A + \partial^{m-j} \cdot B = \partial^{m-j}B$. So the $(n+j)$-th row of S has to consist of the coefficients of $\partial^{m-j}B$. □

Definition 3.8. Let A, B be linear differential operators in $R[\partial]$ of $\mathrm{order}(A) = m, \mathrm{order}(B) = n$, with $m, n > 0$.
By $\partial\mathrm{syl}(A, B)$ we denote the *(differential) Sylvester matrix*; i.e., the $(m+n) \times (m+n)$-matrix whose rows contain the coefficients of

$$\partial^{n-1}A, \ldots, \partial A,\ A, \partial^{m-1}B, \ldots, \partial B,\ B\ .$$

The *(differential Sylvester) resultant* of A and B, $\partial\mathrm{res}(A, B)$, is the determinant of $\partial\mathrm{syl}(A, B)$. $\square$

From Theorems 3.3 and 3.4 the following analogue of Propositions 2.2 and 2.3 is immediate.

Theorem 3.9. *Assume that K has characteristic 0 and that its field of constants is algebraically closed. Let A, B be linear differential operators over R of positive orders. Then the condition $\partial\mathrm{res}(A, B) = 0$ is both necessary and sufficient for there to exist a common non-trivial root of A and B in an extension of K.*

Example: Consider the operators

$$A = 2x - (x^2 + \frac{1}{6})\partial \ , \quad B = (6x^2 + 2) - (3x^3 + \frac{1}{2}x)\partial - (x^2 + \frac{1}{6})\partial^2 \ .$$

These operators correspond to the differential equations

$$2x \cdot y - (x^2 + \frac{1}{6}) \cdot y' = 0, \quad (6x^2 + 2) \cdot y - (3x^3 + \frac{1}{2}x) \cdot y' - (x^2 + \frac{1}{6}) \cdot y'' = 0 \ .$$

$$\partial \mathrm{res}(A, B) = \det \begin{pmatrix} \partial A \\ A \\ B \end{pmatrix} = \det \begin{pmatrix} 2 & 0 & -x^2 - \frac{1}{6} \\ 2x & -x^2 - \frac{1}{6} & 0 \\ 6x^2 + 2 & -3x^3 - \frac{1}{2}x & -x^2 - \frac{1}{6} \end{pmatrix} = 0 \ .$$

So A and B have a common factor on the right of positive order; in fact, this factor is A, and $B = (3x + \partial) \cdot A$.
This also means, that A and B (or their corresponding differential equations) have a common root, namely $\eta = 3x^2 + \frac{1}{2}$.

We close this subsection by stating an analogue of Proposition 2.4.

Theorem 3.10. *Let $A, B \in R[\partial]$. The resultant of A and B is contained in (A, B), the ideal generated by A and B in $R[\partial]$. Moreover, $\partial\mathrm{res}(A, B)$ can be written as a linear combination $\partial\mathrm{res}(A, B) = CA + DB$, with $\mathrm{order}(C) < \mathrm{order}(B)$, and $\mathrm{order}(D) < \mathrm{order}(A)$.*

Proof: Let $S := \partial\mathrm{syl}(A, B)$. Now proceed as in the proof of Proposition 2.4; only instead of multiplying the i-th column of S by x^{m+n-i}, multiply it by ∂^{m+n-i} from the right and add to the last column. This will result in a new matrix T, having the same determinant as S. The columns of T are the same as the corresponding columns of S, except for the last column, which consists of the operators

$$\partial^{n-1}A, \ldots, \partial A,\ A, \partial^{m-1}B, \ldots, \partial B,\ B\ .$$

Expanding the determinant of T w.r.t. its last column, we obtain operators C and D s.t.

$$\partial\mathrm{res}(A, B) = CA + DB,$$

and $\mathrm{order}(C) < \mathrm{order}(B)$, $\mathrm{order}(D) < \mathrm{order}(A)$. $\square$

From Theorem 3.10 we readily obtain an alternative proof that $\partial \mathrm{res}(A, B) = 0$ is a necessary condition for the existence of a non-trivial common root of A and B in an extension of K. The details are left as an exercise for the reader.

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