## Chapter 4 <br> Differential Resultants

The material for this chapter is taken from the technical report
Scott McCallum (Macquarie Univ. Sydney), Franz Winkler (J.Kepler Univ. Linz)
Resultants: Algebraic and Differential
(2018)

## 1 Introduction

This report summarises ongoing discussions of the authors on the topic of differential resultants which have three goals in mind. First, we aim to try to understand existing literature on the topic. Second, we wish to formulate some interesting questions and research goals based on our understanding of the literature. Third, we would like to advance the subject in one or more directions, by pursuing some of these questions and research goals. Both authors have somewhat more background in nondifferential, as distinct from differential, computational algebra. For this reason, our approach to learning about differential resultants has started with a careful review of the corresponding theory of resultants in the purely algebraic (polynomial) case. We try, as far as possible, to adapt and extend our knowledge of purely algebraic resultants to the differential case. Overall, we have the hope of helping to clarify, unify and further develop the computational theory of differential resultants.

There are interesting notions of a differential polynomial resultant in the literature. At first glance it could appear that these notions differ in essential ways. For example, Zwillinger [9 suggested that the concept of a differential resultant of a system of two coupled algebraic ordinary differential equations (AODEs) for $(y(x), z(x))$ (where $x$ is the independent variable and $y$ and $z$ are the dependent variables) could be developed. Such a differential resultant would be a single AODE for $z(x)$ only. While that author sketches how such differential elimination could work for a specific example, no general method is presented. Chardin [3] presented an elegant treatment of resultants and subresultants of (noncommutative) ordinary differential operators. Carra'-Ferro (see for example [1, 2]) published several works on differential resultants of various kinds, with firm algebraic foundations, but the relations to Zwillinger's suggested notion and Chardin's theory might not be immediately clear from glancing through these works.

In fact our study of the subject has revealed to us that the approaches of all three authors mentioned above are intimately related. It would appear that the common source for the essential basic notion of differential resultant can be traced to work of Ritt [8] in the 1930s. After reviewing relevant background material on algebra, both classical and
differential, in Section 2, we will present in Section 3 the simplest case of the differential resultant originally proposed by Ritt: namely, that of two linear homogeneous ordinary differential polynomials over a differential ring or field. Chardin's theory is most closely associated with this simple special case. In Section 4 we will review the algebraic theory of the multipolynomial resultant of Macaulay. In Section 5, using the concepts and results of Section 4, we extend the concept of Section 3 to that of two arbitrary ordinary differential polynomials over a differential field or ring. This could be viewed as a simpler and more streamlined account of Carra'-Ferro's theory. We will see that this theory can be applied to the problem of differential elimination, thereby providing a systematic treatment of the approach suggested by Zwillinger. In Section 6 we survey briefly the work post that of Carra'-Ferro, and in the final section we pose questions for investigation.

## 2 Elementary background material

Let $R$ be a commutative ring with identity element 1 . In the first subsection we shall review the definition of the classical Sylvester resultant res $(f, g)$ of $f(x), g(x) \in R[x]$. We shall state the requirements on $R$ so that $\operatorname{res}(f, g)=0$ is a necessary and sufficient condition for the existence in some extension of $R$ of a solution $\alpha$ to the system

$$
f(x)=g(x)=0
$$

In the second subsection we review elementary differential algebra on $R$. In particular we define the notion of a derivation on $R$, and introduce the ring of differential polynomials over $R$. The elementary background concepts from this section will provide the foundation for the theory of the differential Sylvester resultant, developed in the next section.

### 2.1 Sylvester resultant

In this subsection we review the basic theory of the Sylvester resultant for algebraic polyomials, with an emphasis on the necessary requirements for the underlying coefficient domain. For convenience we assume at the outset that $R$ is an integral domain (commutative ring with 1 , and no zero divisors), and that $K$ is its quotient field. Some results we will state require merely that $R$ be a commutative ring with 1 , and others require a stronger hypothesis, as we shall remark.

Let

$$
f(x)=\sum_{i=0}^{m} a_{i} x^{i}, \quad g(x)=\sum_{j=0}^{n} b_{j} x^{j}
$$

be polynomials of positive degrees $m$ and $n$, respectively, in $R[x]$. If $f$ and $g$ have a common factor $d(x)$ of positive degree, then they have a common root in the algebraic closure $\bar{K}$ of $K$; so the system of equations

$$
\begin{equation*}
f(x)=g(x)=0 \tag{1}
\end{equation*}
$$

has a solution in $\bar{K}$.
On the other hand, if $\alpha \in \bar{K}$ is a common root of $f$ and $g$, then $\operatorname{norm}_{K(\alpha): K}(x-\alpha)$ is a common divisor of $f$ and $g$ in $K[x]$. So, by Gauss' Lemma (for which we need $R$ to be a unique factorization domain) on primitive polynomials there is a similar (only differing by a factor in $K$ ) common factor of $f$ and $g$ in $R[x]$. We summarize these observations as follows:

Proposition 2.1. Let $R$ be a unique factorization domain (UFD) with quotient field $K$. For polynomials $f(x), g(x) \in R[x]$ the following are equivalent:
(i) $f$ and $g$ have a common solution in $\bar{K}$, the algebraic closure of $K$,
(ii) $f$ and $g$ have a common factor of positive degree in $R[x]$.

So now we want to determine a necessary condition for $f$ and $g$ to have a common divisor of positive degree in $R[x]$. Suppose that $f$ and $g$ indeed have a common divisor $d(x)$ of positive degree in $R[x]$; i.e.,

$$
\begin{equation*}
f(x)=d(x) \bar{f}(x), \quad g(x)=d(x) \bar{g}(x) \tag{2}
\end{equation*}
$$

Then for $p(x):=\bar{g}(x), q(x):=-\bar{f}(x)$ we have

$$
\begin{equation*}
p(x) f(x)+q(x) g(x)=0 \tag{3}
\end{equation*}
$$

So there are non-zero polynomials $p$ and $q$ with $\operatorname{deg} p<\operatorname{deg} g, \operatorname{deg} q<\operatorname{deg} f$, satisfying equation (3). This means that the linear system

$$
\left(\begin{array}{llllll}
p_{n-1} & \cdots & p_{0} & q_{m-1} & \cdots & q_{0}
\end{array}\right) \cdot\left(\begin{array}{c}
A  \tag{4}\\
\cdots \\
B
\end{array}\right)=0
$$

where

$$
\begin{gathered}
A=\left(\begin{array}{cccccccc}
a_{m} & a_{m-1} & \cdots & a_{0} & & & \\
& a_{m} & a_{m-1} & \cdots & a_{0} & & \\
& & \ddots & & & \ddots & \\
& & & a_{m} & a_{m-1} & \cdots & a_{0}
\end{array}\right) \in R_{m+n}^{n}, \\
B=\left(\begin{array}{cccccccc}
b_{n} & b_{n-1} & \cdots & b_{0} & & & \\
& b_{n} & b_{n-1} & \cdots & b_{0} & & \\
& & \ddots & & & \ddots & \\
& & & b_{n} & b_{n-1} & \cdots & b_{0}
\end{array}\right) \in R_{m+n}^{m}
\end{gathered}
$$

has a non-trivial solution. The matrix of this system (4) is called the Sylvester matrix of $f$ and $g$. Thus, the determinant of the Sylvester matrix of $f$ and $g$ is 0 . The resultant of $f$ and $g$, $\operatorname{res}(f, g)$, is this determinant, and it is clear that the resultant is a polynomial expression of the coefficients of $f$ and $g$, and therefore an element of the integral domain $R$. This does not require $R$ to be a UFD. Summarizing:

Proposition 2.2. Let $f, g \in R[x]$, for $R$ an integral domain.
$\operatorname{res}(f, g)=0$ is a necessary condition for $f$ and $g$ to have a common factor of positive degree.

If we identify a polynomial of degree $d$ with the vector of its coefficients of length $d+1$, we may also express this in terms of the linear map

$$
\begin{aligned}
S: & K^{m+n} & \longrightarrow & K^{m+n} \\
& \left(p_{n-1}, \ldots, p_{0}, q_{m-1}, \ldots, q_{0}\right) & \mapsto & \text { coefficients of } p f+q g
\end{aligned}
$$

Obviously the existence of a non-trivial linear combination (3) is equivalent to $S$ having a non-trivial kernel, and therefore to $S$ having determinant 0 .

But is the vanishing of the resultant also a sufficient condition for $f$ and $g$ to have a common factor of positive degree? Suppose that $\operatorname{res}(f, g)=0$. This means that (3) has a non-trivial solution $u(x), v(x)$ of bounded degree; so

$$
u(x) f(x)=-v(x) g(x)
$$

These co-factors will be in $K[x]$, but of course we can clear denominators and have a similar relation with co-factors in $R[x]$. If we now require the coefficient domain $R$ to be a unique factorization domain (UFD), we see that every irreducible factor of $f$ must appear on the right hand side with at least the same multiplicity. Not all of these factors can
be subsumed in $v$, because $v$ is of lower degree than $f$. So at least one of the irreducible factors of $f$ must divide $g$. Thus we have:

Proposition 2.3. Let $f, g \in R[x]$, for $R$ a UFD.
$\operatorname{res}(f, g)=0$ is also a sufficient condition for $f$ and $g$ to have a common factor of positive degree; and therefore a common solution in $\bar{K}$.

A further property is of interest and importance.
Proposition 2.4. The resultant is a constant in the ideal generated by $f$ and $g$ in $R[x]$; i.e. we can write

$$
\begin{equation*}
\operatorname{res}(f, g)=u(x) f(x)+v(x) g(x) \tag{5}
\end{equation*}
$$

with $u, v \in R[x]$. Moreover, these cofactors satisfy the degree bounds $\operatorname{deg}(u)<\operatorname{deg}(g)$, $\operatorname{deg}(v)<\operatorname{deg}(f)$.

Proof: We follow an argument given in [4]. In fact, Collins proves this fact for $R$ being a general commutative ring with 1 .

Consider the Sylvester matrix $S=(A \vdots B)^{T}$; i.e. the $(m+n) \times(m+n)$ matrix, whose first $n$ rows consist of the coefficients of

$$
x^{n-1} \cdot f(x), \ldots, x \cdot f(x), f(x)
$$

and whose last $m$ rows consist of the coefficients of

$$
x^{m-1} \cdot g(x), \ldots, x \cdot g(x), g(x)
$$

Now, for $1 \leq i<m+n$, multiply the $i$ th column of $S$ by $x^{m+n-i}$ and add to the last column. This will result in a new matrix $T$, having the same determinant as $S$. The columns of $T$ are the same as the corresponding columns of $S$, except for the last column, which consists of the polynomials

$$
x^{n-1} \cdot f(x), \ldots, x \cdot f(x), f(x), x^{m-1} \cdot g(x), \ldots, x \cdot g(x), g(x)
$$

Expanding the determinant of $T$ w.r.t. its last column, we obtain polynomials $u(x)$ and $v(x)$ satisfying the relation (5), and also the degree bounds.

An alternative approach (similar to that above but with a slightly different emphasis) to defining the Sylvester resultant of $f(x)$ and $g(x)$ is to regard all the coefficients $a_{i}$ and $b_{j}$ of $f$ and $g$ as distinct and unrelated indeterminates. The indeterminates $a_{m}$ and $b_{n}$ are then referred to as the formal leading coefficients of $f$ and $g$, respectively. In effect we take $R$ to be the domain $\mathbb{Z}\left[a_{m}, \ldots, a_{0}, b_{n}, \ldots, b_{0}\right]$. This approach allows us to study the resultant $\operatorname{res}(f, g)$ as a polynomial in the $m+n+2$ indeterminates $a_{i}$ and $b_{j}$. Indeed it is not hard to see that $\operatorname{res}(f, g)$ is homogeneous in the $a_{i}$ of degree $n$, homogeneous in the $b_{j}$ of degree $m$, and has the "principal term" $a_{m}^{n} b_{0}^{m}$ (from the principal diagonal). With this approach, adjustment of some of the basic facts is needed. For example, the analogue of Proposition 2.3 would state that, for $D$ a UFD, after replacement of all the coefficients $a_{i}$ and $b_{j}$ by elements of $D, \operatorname{res}(f, g)=0$ is a sufficient condition for either $f(x)$ and $g(x)$ to have a common factor of positive degree, or $a_{m}=b_{n}=0$.

Another variation on defining the Sylvester resultant of two polynomials is to start instead with two homogeneous polynomials $F(x, y)=\sum_{i=0}^{m} a_{i} x^{i} y^{m-i}$ and $G(x, y)=$
$\sum_{j=0} b_{j} x^{j} y^{n-j}$. Let us similarly regard the coefficients $a_{i}$ and $b_{j}$ as indeterminates. Then the resultant of $F$ and $G$ is defined as $\operatorname{res}(F, G)=\operatorname{res}(f, g)$, where $f(x)=F(x, 1)$ and $g(x)=G(x, 1)$. Our analogue of Proposition 2.3 then becomes simpler. Combining it with homogeneous analogues of Propositions 2.1 and 2.2 we have:

Proposition 2.5. After assigning values to the coefficients from a $U F D D, \operatorname{res}(F, G)=0$ is a necessary and sufficient condition for $F(x, y)$ and $G(x, y)$ to have a common factor of positive degree over $D$, hence for a common zero to exist over an extension of the quotient field of $D$.

### 2.2 Basic differential algebra

Let $R$ be a commutative ring with 1 . A derivation on $R$ is a mapping $\partial: R \rightarrow R$ such that $\partial(a+b)=\partial(a)+\partial(b)$ and $\partial(a b)=\partial(a) b+a \partial(b)$ for all $a, b \in R$. That $\partial(0)=0$ and $\partial(1)=0$ follow readily from these axioms. We sometimes denote the derivative of $a \partial(a)$ by $a^{\prime}$. Such a ring (or integral domain or field) $R$ together with a derivation on $R$ is called a differential ring (or integral domain or field, respectively). In such a ring $R$ elements $r$ such that $r^{\prime}=0$ are known as constants and the set $C$ of constants comprises a subring of $R$. If $R$ is a field, $C$ is a subfield of $R$. An ideal $I$ of such a ring $R$ is known as a differential ideal if $r \in I$ implies $r^{\prime} \in I$. If $r_{1}, \ldots, r_{n} \in R$ we denote by $\left\langle r_{1}, \ldots, r_{n}\right\rangle$ the differential ideal generated by $r_{1}, \ldots, r_{n}$, that is, the ideal generated by the $r_{i}$ and all their derivatives.

Example 2.6. The familiar rings such as $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are differential rings if we set $\partial(a)=0$ for all elements $a$.

Example 2.7. Let $K$ be a field and $t$ an indeterminate over $K$. Then $K[t]$, equipped with the derivation $\partial=d / d t$, is a differential integral domain and its quotient field $K(t)$ is a differential field, again with standard differentiation as its derivation. $K$ is the ring (field) of constants of $K[t](K(t))$.
Example 2.8. Let $(R, \partial)$ be a differential ring. Let $x=x^{(0)}, x^{(1)}, x^{(2)}, \ldots$ be distinct indeterminates over $R$. Put $\partial\left(x^{(i)}\right)=x^{(i+1)}$ for all $i \geq 0$. Then $\partial$ can be extended to a derivation on the polynomial ring $R\{x\}:=R\left[x^{(0)}, x^{(1)}, \ldots\right]$ in a natural way, and we denote this extension also by $\partial$. The ring $R\{x\}$ together with this extended $\partial$ is a differential ring, called the ring of differential polynomials in the differential indeterminate $x$ over $R$. An element $f(x)=\sum_{i=0}^{m} a_{i} x^{(i)}$ of $R\{x\}$ with $a_{m} \neq 0$ has order $m$ and leading coefficient $a_{m}$. (Remark. It may be helpful to think of elements of $R$ and of $x, x^{(1)}, \ldots$ as functions of an indeterminate $t$, and to regard $\partial$ as differentiation with respect to t.) If $(K, \partial)$ is a differential field then $K\{x\}$ is a differential integral domain, and its derivation extends uniquely to the quotient field. We write $K\langle x\rangle$ for this quotient field; its elements are differential rational functions of $x$ over $K$.

Let $(R, \partial)$ be a differential integral domain. An ultimate aim of this paper is to define and study a certain resultant of two elements of the differential ring (indeed domain) $R\{x\}$ introduced above. In the next section, we shall consider a simple and important $R$-submodule of $R\{x\}$ (considered as left $R$-module), namely that which comprises those elements of $R\{x\}$ which are linear and homogeneous. For two such elements we will introduce an analogue of the classical Sylvester resultant reviewed in Section 2.

## 3 Differential Sylvester resultant

Let $(R, \partial)$ be a differential integral domain. Recall from Section 2 that the ring (indeed domain) of differential polynomials in the differential indeterminate $x$ is denoted by $R\{x\}$. Then $R\{x\}$ is also a (left) $R$-module, and we denote by $R_{L H}\{x\}$ the $R$-submodule comprising those elements of $R\{x\}$ which are linear and homogeneous. We aim in this section to define a certain resultant, known as a differential Sylvester resultant, of two elements of $R_{L H}\{x\}$. We shall begin by studying a closely related noncommutative ring: namely, we consider the ring $R[\partial]$ of linear differential operators on $R$. As we shall see, there is an important relationship between $R[\partial]$, considered as left $R$-module, and $R_{L H}\{x\}$ : these are isomorphic as left $R$-modules. Thus the differential theory of $R[\partial]$ and $R_{L H}\{x\}$ can to an extent be developed in parallel. The details are provided in the next two subsections.

### 3.1 Resultant of two linear differential operators

This subsection follows the presentation of [3], and elaborates on a number of points from that source. Let $(R, \partial)$ be a differential integral domain. Recall that we sometimes denote $\partial(a)$ by $a^{\prime}$. Then $K$, the quotient field of $R$, is naturally equipped with an extension of this derivation, which we will also denote by $\partial$ (and sometimes by ${ }^{\prime}$ ).

We consider the ring of linear differential operators $R[\partial]$, where the application of $A=\sum_{i=0}^{m} a_{i} \partial^{i}$ to $r \in R$ is defined as

$$
A(r)=\sum_{i=0}^{m} a_{i} r^{(i)}
$$

Here $r^{(i)}$ denotes the $i$-fold application of $\partial$ (that is, ') to $r$. If $a_{m} \neq 0$, the order of $A$ is $m$ and $a_{m}$ is the leading coefficient of $A$. Now the application of $A$ can naturally be extended to $K$, and to any extension of $K$. If $A(\eta)=0$, with $\eta$ in $R, K$ or any extension of $K$, we call $\eta$ a root of the linear differential operator $A$.

The application of the constant operator $r$ to $a$ yields $r(a)=r \cdot a^{(0)}=r \cdot a$.
The ring $R[\partial]$ is non-commutative; let us see what the commutation rule is. If we apply $\partial r$ to $a$ we get

$$
\partial r(a)=\partial(r a)=r \partial(a)+(\partial(r)) a=r \partial(a)+r^{\prime} a=\left(r \partial+r^{\prime}\right)(a) .
$$

So the corresponding rule for the multiplication of $\partial$ by an element of $r \in R$ is

$$
\partial r=r \partial+r^{\prime}
$$

Note that $\partial r$, which denotes the operator product of $\partial$ and $r$, is distinct from $\partial(r)$ (that is, from $r^{\prime}$ ), the application of map $\partial$ to $r$.
Proposition 3.1. For $n \in N: \partial^{n} r=\sum_{i=0}^{n}\binom{n}{i} r^{(n-i)} \partial^{i}$.
Proof: For $n=0$ this obviously holds.

Assume the fact holds for some $n \in \mathbb{N}$. Then

$$
\begin{aligned}
\partial^{n+1} r & =\partial\left(\partial^{n} r\right)=\partial\left(\sum_{i=0}^{n}\binom{n}{i} r^{(n-i)} \partial^{i}\right) \\
& =\sum_{i=0}^{n}\binom{n}{i} \partial r^{(n-i)} \partial^{i}=\sum_{i=0}^{n}\binom{n}{i}\left[r^{(n-i)} \partial+r^{(n-i+1)}\right] \partial^{i} \\
& =\sum_{i=0}^{n}\binom{n}{i} r^{(n-i)} \partial^{i+1}+\sum_{i=0}^{n}\binom{n}{i} r^{(n-i+1)} \partial^{i} \\
& =\sum_{n=1}^{n+1}\binom{n}{i-1} r^{(n+1-i)} \partial^{i}+\sum_{i=0}^{n}\binom{n}{i} r^{(n-i+1)} \partial^{i} \\
& =\binom{n}{n} r^{(0)} \partial^{n+1}+\sum_{i=1}^{n}\left[\binom{n}{i-1}+\binom{n}{i}\right] r^{(n+1-i)} \partial^{i}+\binom{n}{0} r^{(n+1)} \partial^{0} \\
& =\sum_{i=0}^{n+1}\binom{n+1}{i} r^{(n+1-i)} \partial^{i} .
\end{aligned}
$$

From a linear homogeneous ODE $p(x)=0$, with $p(x) \in R\{x\}$,

$$
p(x)=p_{0}(t) x+p_{1}(t) x^{\prime}+\cdots+p_{n}(t) x^{(n)}=0
$$

we can extract a linear differential operator

$$
\mathcal{O}(p)=A=\sum_{i=0}^{n} p_{i} \partial^{i}
$$

such that the given ODE can be written as

$$
A(x)=0,
$$

in which $x$ is regarded as an unknown element of $R$, $K$ or some extension of $K$. Such a linear homogeneous ODE always has the trivial solution $x=0$; so a linear differential operator always has the trivial root 0 .

In [3] it is stated that $K[\partial]$ is left-Euclidean, and a few brief remarks are provided by way of proof. Since the concept of a left-Euclidean ring is not as widely known as that of Euclidean ring, it may be helpful to recall its definition here. A ring $R$ is left-Euclidean if there exists a function $d: R-\{0\} \rightarrow \mathbb{N}$ such that for all $A, B$ in $R$, with $B \neq 0$, there exist $Q$ and $R$ in $R$ such that $A=Q B+R$, with $d(R)<d(B)$ or $R=0$. If one wishes to provide a complete proof of the claim that $K[\partial]$ is left-Euclidean (in which we take $d(A)$ to be the order of $A$ ), Proposition 3.1 above is useful. For example, by way of proof hint, Chardin claims that the operator $A-(a / b) \partial^{m-n} B$ is of order less than $m$, where $a$ and $b$ are the leading coefficients of $A$ and $B$, respectively, and $m$ and $n$ are their orders, with $m \geq n$ assumed. To show this claim, it suffices to show that the term $(a / b) \partial^{m-n} B$ consists of $a \partial^{m}$ plus terms of order less than $m$. This follows by applications of Proposition 3.1, putting $n=m-n$ and $r$ equal to each coeficient of operator $B$ in turn.

It follows from the left-Euclidean property that every left-ideal ${ }_{K} I$ of the form ${ }_{K} I=$ $(A, B)$ is principle, and is generated by the right-gcd of $A$ and $B$. As remarked in [3] with reference to [5], under suitable assumptions on $K$, any linear differential operator of positive order has a root in some extension of $K$. We state this result precisely.
Theorem 3.2. (Ritt-Kolchin). Assume that the differential field $K$ has characteristic 0 and that its field $C$ of constants is algebraically closed. Then, for any linear differential operator $A$ over $K$ of positive order $n$, there exist $n$ roots $\eta_{1}, \ldots, \eta_{n}$ in a suitable extension of $K$, such that the $\eta_{i}$ are linearly independent over $C$. Moreover, the field $K\left\langle\eta_{1}, \ldots, \eta_{n}\right\rangle$ contains no constant not in $C$.

This result is stated and proved in [7] using results from [6] and 8]. The field $K\left\langle\eta_{1}, \ldots, \eta_{n}\right\rangle$ associated with $A$ is known as a Picard-Vessiot extension of $K$ (for $A$ ). Henceforth assume the hypotheses of Theorem 3.2.

It follows from Theorem 3.2 that if the operators $A, B \in K[\partial]$ have a common factor $F$ of positive order on the right, i.e.,

$$
\begin{equation*}
A=\bar{A} \cdot F, \quad \text { and } B=\bar{B} \cdot F \tag{6}
\end{equation*}
$$

then they have a non-trivial common root in a suitable extension of $K$. For by Theorem 3.2, $F$ has a root $\eta \neq 0$ in an extension of $K$. We have $A(\eta)=\bar{A}(F(\eta))=\bar{A}(0)=0$ and similarly $B(\eta)=0$.

On the other hand, if $A$ and $B$ have a non-trivial common root $\eta$ in a suitable extension of $K$, we show that they have a common right factor of positive order in $K[\partial]$. Let $F$ be a nonzero differential operator of lowest order s.t. $F(\eta)=0$. Then $F$ has positive order. Because the ring of operators is left-Euclidean, $F$ is unique up to multiplication of non-zero elements of $K$. This $F$ is a right divisior of both $A$ and $B$. To see this, apply division in the left-Euclidean ring $K[\partial]$ :

$$
A=Q \cdot F+R
$$

with the order of $R$ less than the order of $F$, or $R=0$. Apply both sides of this equation to $\eta$ :

$$
A(\eta)=(Q \cdot F)(\eta)+R(\eta)
$$

Since $A(\eta)=0$ and $F(\eta)=0, R(\eta)=0$. Therefore, by minimality of $F, R=0$. Hence $F$ is a right divisor of $A$. We see that $F$ is a right divisor of $B$ similarly. We summarize our result in the following theorem, which is the closest analogue of Proposition 2.1 we can state.

Theorem 3.3. Assume that $K$ has characteristic 0 and that its field of constants is algebraically closed. Let $A, B$ be differential operators of positive orders in $K[\partial]$. Then the following are equivalent:
(i) $A$ and $B$ have a common non-trivial root in an extension of $K$,
(ii) $A$ and $B$ have a common factor of positive order on the right in $K[\partial]$.

Now let us see that the existence of a non-trivial factor (6) is equivalent to the existence of a non-trivial order-bounded linear combination

$$
\begin{equation*}
C A+D B=0 \tag{7}
\end{equation*}
$$

with $\operatorname{order}(C)<\operatorname{order}(B)$ and $\operatorname{order}(D)<\operatorname{order}(A)$, and $(C, D) \neq(0,0)$.
For given $A, B \in K[\partial]$, with $m=\operatorname{order}(A), n=\operatorname{order}(B)$, consider the linear map

$$
\begin{array}{rlrl}
S: & K^{m+n} & \longrightarrow K^{m+n} \\
& \left(c_{n-1}, \ldots, c_{0}, d_{m-1}, \ldots, d_{0}\right) & \mapsto & \text { coefficients of } C A+D B \tag{8}
\end{array}
$$

Obviously the existence of a non-trivial linear combination (7) is equivalent to $S$ having a non-trivial kernel, and therefore to $S$ having determinant 0 . Indeed we have the following result.

Theorem 3.4. $\operatorname{det}(S)=0$ if and only if $A$ and $B$ have a common factor (on the right) in $K[\partial]$ of positive order.

Proof: Suppose $\operatorname{det}(S)=0$. This means that $S$ cannot be surjective. Now the right-gcd $G$ of $A$ and $B$ can be written as an order-bounded linear combination of $A$ and $B$, so it is in the image of the map $S$. This means that $G$ cannot be trivial (that is, $G$ cannot be an element of $K$ ), because otherwise $S$ would be surjective.

On the other hand, suppose that $\operatorname{det}(S) \neq 0$. Then the linear map is invertible; in particular, it is surjective. Therefore there exist $C, D \in K[\partial]$ with appropriate degree bounds, s.t. $1=C A+D B$. So every common divisor (on the right) of $A$ and $B$ is a common divisor of 1 . Therefore no common divisor of $A$ and $B$ could have positive order.

So let us see which linear conditions on the coefficients of $A$ and $B$ we get by requiring that (7) has a non-trivial solution of bounded order, i.e.,

$$
\operatorname{order}(C)<\operatorname{order}(B) \quad \text { and } \quad \operatorname{order}(D)<\operatorname{order}(A)
$$

Example 3.5. $\operatorname{order}(A)=2=\operatorname{order}(B)$

$$
\left(c_{1} \partial+c_{0}\right)\left(a_{2} \partial^{2}+a_{1} \partial+a_{0}\right)+\left(d_{1} \partial+d_{0}\right)\left(b_{2} \partial^{2}+b_{1} \partial+b_{0}\right)
$$

order 3:

$$
\begin{aligned}
& c_{1} \partial a_{2} \partial^{2}=c_{1}\left(a_{2} \partial+a_{2}^{\prime}\right) \partial^{2}=c_{1} a_{2} \partial^{3}+c_{1} a_{2}^{\prime} \partial^{2} \\
& d_{1} \partial b_{2} \partial^{2}=d_{1}\left(b_{2} \partial+b_{2}^{\prime}\right) \partial^{2}=d_{1} b_{2} \partial^{3}+d_{1} b_{2}^{\prime} \partial^{2}
\end{aligned}
$$

order 2:

$$
c_{1} a_{2}^{\prime} \partial^{2}(\text { from above })+c_{1} \partial a_{1} \partial+c_{0} a_{2} \partial^{2}=c_{1} a_{2}^{\prime} \partial^{2}+c_{1} a_{1} \partial^{2}+c_{0} a_{2} \partial^{2}+c_{1} a_{1}^{\prime} \partial
$$

analogous for $b$ and $d$
order 1:

$$
c_{1} a_{1}^{\prime} \partial \text { (from above) }+c_{1} \partial a_{0}+c_{0} a_{1} \partial=c_{1} a_{1}^{\prime} \partial+c_{1}\left(a_{0} \partial+a_{0}^{\prime}\right)+c_{0} a_{1} \partial=c_{1} a_{1}^{\prime} \partial+c_{1} a_{0} \partial+c_{0} a_{1} \partial+c_{1} a_{0}^{\prime}
$$

analogous for $b$ and $d$
order 0:

$$
\begin{gathered}
c_{1} a_{0}^{\prime}(\text { from above })+c_{0} a_{0} \\
\text { analogous for } b \text { and } d
\end{gathered}
$$

So, finally,

$$
\left(\begin{array}{llll}
c_{1} & c_{0} & d_{1} & d_{0}
\end{array}\right) \cdot\left(\begin{array}{cccc}
a_{2} & a_{1}+a_{2}^{\prime} & a_{0}+a_{1}^{\prime} & a_{0}^{\prime} \\
0 & a_{2} & a_{1} & a_{0} \\
b_{2} & b_{1}+b_{2}^{\prime} & b_{0}+b_{1}^{\prime} & b_{0}^{\prime} \\
0 & b_{2} & b_{1} & b_{0}
\end{array}\right)=\left(\begin{array}{llll}
0 & 0 & 0 & 0
\end{array}\right) .
$$

Observe, that the rows of this matrix consist of the coefficients of

$$
\partial A, A, \partial B, B
$$

Comparing this to the example in [3], p.3, we see that after interchanging of rows this is the same matrix.

Example 3.6. $\operatorname{order}(A)=2, \operatorname{order}(B)=3$

$$
\left(c_{2} \partial+c_{1} \partial+c_{0}\right)\left(a_{2} \partial^{2}+a_{1} \partial+a_{0}\right)+\left(d_{1} \partial+d_{0}\right)\left(b_{3} \partial^{3}+b_{2} \partial^{2}+b_{1} \partial+b_{0}\right)
$$

order 4:

$$
\begin{gathered}
c_{2} \partial^{2} a_{2} \partial^{2}+d_{1} \partial b_{3} \partial^{3}=0 \\
\underline{a_{2} c_{2} \partial^{4}}+2 a_{2}^{\prime} c_{2} \partial^{3}+a_{2}^{\prime \prime} c_{2} \partial^{2}+\underline{b_{3} d_{1} \partial^{4}}+b_{3}^{\prime} d_{1} \partial^{3}=0
\end{gathered}
$$

order 3:

$$
\begin{aligned}
& \left(2 a_{2}^{\prime} c_{2} \partial^{3}+a_{2}^{\prime \prime} c_{2} \partial^{2} \text { from above }\right)+c_{2} \partial^{2} a_{1} \partial+c_{1} \partial a_{2} \partial^{2}+\left(b_{3}^{\prime} d_{1} \partial^{3} \text { from above }\right)+d_{1} \partial b_{2} \partial^{2}+d_{0} b_{3} \partial^{3}=0 \\
& \underline{2 a_{2}^{\prime} c_{2} \partial^{3}}+\underline{a_{1} c_{2} \partial^{3}}+\underline{a_{2} c_{1} \partial^{3}}+a_{2}^{\prime \prime} c_{2} \partial^{2}+2 a_{1}^{\prime} c_{2} \partial^{2}+a_{1}^{\prime \prime} c_{2} \partial+\underline{b_{3}^{\prime} d_{1} \partial^{3}}+\underline{b_{2} d_{1} \partial^{3}}+\underline{b_{3} d_{0} \partial^{3}}+b_{2}^{\prime} d_{1} \partial^{2}=0
\end{aligned}
$$

order 2:

$$
\begin{array}{r}
\left(a_{2}^{\prime \prime} c_{2} \partial^{2}+2 a_{1}^{\prime} c_{2} \partial^{2}+a_{1}^{\prime \prime} c_{2} \partial+a_{2}^{\prime} c_{1} \partial^{2} \text { from above }\right)+c_{2} \partial^{2} a_{0}+c_{1} \partial a_{1} \partial+c_{0} a_{2} \partial^{2} \\
+\left(b_{2}^{\prime} d_{1} \partial^{2} \text { from above }\right)+d_{1} \partial b_{1} \partial+d_{0} b_{2} \partial^{2}=0 \\
\underline{a_{2}^{\prime \prime} c_{2} \partial^{2}}+\underline{2 a_{1}^{\prime} c_{2} \partial^{2}}+a_{1}^{\prime \prime} c_{2} \partial+\underline{a_{2}^{\prime} c_{1} \partial^{2}}+\underline{a_{0} c_{2} \partial^{2}}+2 a_{0}^{\prime} c_{2} \partial+a_{0}^{\prime \prime} c_{2}+\underline{a_{1} c_{1} \partial^{2}}+a_{1}^{\prime} c_{1} \partial+\underline{a_{2} c_{0} \partial^{2}} \\
+\underline{b_{2}^{\prime} d_{1} \partial^{2}}+\underline{b_{1} d_{1} \partial^{2}}+b_{1}^{\prime} d_{1} \partial+\underline{b_{2} d_{0} \partial^{2}}=0
\end{array}
$$

order 1:
$\left(a_{1}^{\prime \prime} c_{2} \partial+2 a_{0}^{\prime} c_{2} \partial+a_{0}^{\prime \prime} c_{2}+a_{1}^{\prime} c_{1} \partial\right.$ from above $)+c_{1} \partial a_{0}+c_{0} a_{1} \partial+\left(b_{1}^{\prime} d_{1} \partial\right.$ from above $)+d_{1} \partial b_{0}+d_{0} b_{1} \partial=0$

$$
\underline{a_{1}^{\prime \prime} c_{2} \partial}+\underline{2 a_{0}^{\prime} c_{2} \partial}+\underline{a_{1}^{\prime} c_{1} \partial}+\underline{a_{0} c_{1} \partial}+a_{0}^{\prime} c_{1}+\underline{a_{1} c_{0} \partial}+a_{0}^{\prime \prime} c_{2}+\underline{b_{1}^{\prime} d_{1} \partial}+\underline{b_{0} d_{1} \partial}+b_{0}^{\prime} d_{1}+\underline{b_{1} d_{0} \partial}=0
$$

order 0:

$$
\left(\underline{a_{0}^{\prime} c_{1}}+\underline{a_{0}^{\prime \prime} c_{2}} \text { from above }\right)+\underline{a_{0} c_{0}}+\left(\underline{b_{0}^{\prime} d_{1}} \text { from above }\right)+\underline{b_{0} d_{0}}=0 .
$$

So, finally

$$
\left(\begin{array}{lllll}
c_{2} & c_{1} & c_{0} & d_{1} & d_{0}
\end{array}\right) \cdot\left(\begin{array}{ccccc}
a_{2} & a_{1}+2 a_{2}^{\prime} & a_{0}+2 a_{1}^{\prime}+a_{2}^{\prime \prime} & 2 a_{0}^{\prime}+a_{1}^{\prime \prime} & a_{0}^{\prime \prime} \\
0 & a_{2} & a_{1}+a_{2}^{\prime} & a_{0}+a_{1}^{\prime} & a_{0}^{\prime} \\
0 & 0 & a_{2} & a_{1} & a_{0} \\
b_{3} & b_{2}+b_{3}^{\prime} & b_{1}+b_{2}^{\prime} & b_{0}+b_{1}^{\prime} & b_{0}^{\prime} \\
0 & b_{3} & b_{2} & b_{1} & b_{0}
\end{array}\right)=\left(\begin{array}{llll}
0 & \cdots & 0
\end{array}\right) .
$$

Observe, that the rows of this matrix consist of the coefficients of

$$
\partial^{2} A, \partial A, A, \partial B, B
$$

Theorem 3.7. The linear map $S$ in (8) corresponding to (7) is given by the matrix whose rows are $\partial^{n-1} A, \ldots, \partial A, A, \partial^{m-1} B, \ldots, \partial B, B$.

Proof: Let $v=\left(c_{n-1}, \ldots, c_{0}, d_{m-1}, \ldots, d_{0}\right)$.
Consider an index $i$ between 1 and $n$. If $c_{n-i}=1$, and all the other components of $v$ are 0 , then $v$ is mapped by $S$ to $\partial^{n-i} \cdot A+0 \cdot B=\partial^{n-i} A$. So the $i$-th row of $S$ has to consist of the coefficients of $\partial^{n-i} A$.
Consider an index $j$ between 1 and $m$. If $d_{m-j}=1$, and all the other components of $v$ are 0 , then $v$ is mapped by $S$ to $0 \cdot A+\partial^{m-j} \cdot B=\partial^{m-j} B$. So the $(n+j)$-th row of $S$ has to consist of the coefficients of $\partial^{m-j} B$.

Definition 3.8. Let $A, B$ be linear differential operators in $R[\partial]$ of $\operatorname{order}(A)=m$, order $(B)=$ $n$, with $m, n>0$.
By $\partial \operatorname{syl}(A, B)$ we denote the (differential) Sylvester matrix; i.e., the $(m+n) \times(m+n)$ matrix whose rows contain the coefficients of

$$
\partial^{n-1} A, \ldots, \partial A, A, \partial^{m-1} B, \ldots, \partial B, B
$$

The (differential Sylvester) resultant of $A$ and $B, \partial \operatorname{res}(A, B)$, is the determinant of $\partial \operatorname{syl}(A, B)$.

From Theorems 3.3 and 3.4 the following analogue of Propositions 2.2 and 2.3 is immediate.

Theorem 3.9. Assume that $K$ has characteristic 0 and that its field of constants is algebraically closed. Let $A, B$ be linear differential operators over $R$ of positive orders. Then the condition $\partial \operatorname{res}(A, B)=0$ is both necessary and sufficient for there to exist a common non-trivial root of $A$ and $B$ in an extension of $K$.

We close this subsection by stating an analogue of Proposition 2.4.
Theorem 3.10. Let $A, B \in R[\partial]$. The resultant of $A$ and $B$ is contained in $(A, B)$, the ideal generated by $A$ and $B$ in $R[\partial]$. Moreover, $\partial \operatorname{res}(A, B)$ can be written as a linear combination $\partial \operatorname{res}(A, B)=C A+D B$, with $\operatorname{order}(C)<\operatorname{order}(B)$, and $\operatorname{order}(D)<\operatorname{order}(A)$.

Proof: Let $S:=\partial \operatorname{syl}(A, B)$. Now proceed as in the proof of Proposition 2.4; only instead of multiplying the $i$-th column of $S$ by $x^{m+n-i}$, multiply it by $\partial^{m+n-i}$ from the right and add to the last column. This will result in a new matrix $T$, having the same determinant as $S$. The columns of $T$ are the same as the corresponding columns of $S$, except for the last column, which consists of the operators

$$
\partial^{n-1} A, \ldots, \partial A, A, \partial^{m-1} B, \ldots, \partial B, B
$$

Expanding the determinant of $T$ w.r.t. its last column, we obtain operators $C$ and $D$ s.t.

$$
\partial \operatorname{res}(A, B)=C A+D B
$$

and $\operatorname{order}(C)<\operatorname{order}(B)$, order $(D)<\operatorname{order}(A)$.
From Theorem 3.10 we readily obtain an alternative proof that $\partial \operatorname{res}(A, B)=0$ is a necessary condition for the existence of a non-trivial common root of $A$ and $B$ in an extension of $K$. The details are left as an exercise for the reader.

### 3.2 Resultant of two linear homogeneous differential polynomials

The results for differential resultants which we have derived for linear differential operators can also be stated in terms of linear homogeneous differential polynomials. Such a treatment facilitates the generalization to the non-linear algebraic differential case.

Let $(R, \partial)$ be a differential domain with quotient field $K$. Then elements of $R\{x\}$ can be interpreted as algebraic ordinary differential equations (AODEs). For instance, the differential polynomial

$$
3 x x^{(1)}+2 t x^{(2)} \in \mathbb{C}(t)\{x\}
$$

corresponds to the AODE

$$
3 x(t) x^{\prime}(t)+2 t x^{\prime \prime}(t)=0 .
$$

The next proposition says that linear differential operators correspond to linear homogeneous differential polynomials in a natural way. Recall that $R_{L H}\{x\}$ denotes the left $R$-submodule of $R\{x\}$ comprising those elements of $R\{x\}$ which are linear and homogeneous.

Proposition 3.11. $R[\partial]$ and $R_{L H}\{x\}$ are isomporphic as left $R$-modules. $K[\partial]$ and $K_{L H}\{x\}$ are isomporphic as left vector spaces over $K$.

Proof: Define $\mathcal{P}: R[\partial] \rightarrow R_{L H}\{x\}$ as follows. Given $A=\sum_{i=0}^{m} a_{i} \partial^{i}$, let $\mathcal{P}(A)=f(x)$, where $f(x)=\sum_{i=0}^{m} a_{i} x^{(i)}$. ( $\mathcal{P}$ stands for (linear homogeneous differential) polynomial.) Then we can easily verify that $\mathcal{P}$ is an isomorphism of left $R$-modules. The inverse of $\mathcal{P}$ is the mapping $\mathcal{O}: R_{L H\{x\}} \rightarrow R[\partial]$, with $\mathcal{O}(f(x))=A$. ( $\mathcal{O}$ stands for (linear differential) operator.) Note that $\mathcal{P}$ has a natural extension, also denoted by $\mathcal{P}: K[\partial] \rightarrow K_{L H}\{x\} ;$ and likewise for $\mathcal{O}$. The extended $\mathcal{P}$ is an isomorphism of vector spaces over $K$.

Definition 3.12. Let $f(x)$ and $g(x)$ be elements of $R_{L H}\{x\}$ of positive orders $m$ and $n$, respectively. Then the (differential) Sylvester matrix of $f(x)$ and $g(x)$, denoted by $\partial \operatorname{syl}(f, g)$, is $\partial \operatorname{syl}(A, B)$, where $A=\mathcal{O}(f)$ and $B=\mathcal{O}(g)$. The (differential Sylvester) resultant of $f(x)$ and $g(x)$, denoted by $\partial \operatorname{res}(f, g)$, is $\partial \operatorname{res}(A, B)$.

We may observe that the $m+n$ rows of $\partial \operatorname{syl}(f, g)$ contain the coefficients of

$$
f^{(n-1)}(x), \ldots, f^{(1)}(x), f(x), g^{(m-1)}(x), \ldots, g^{(1)}(x), g(x)
$$

The following analogue and slight reformulation of Theorem 3.9 is immediate.
Theorem 3.13. Assume that $K$ has characteristic 0 and that its field of constants is algebraically closed. Let $f(x), g(x)$ be linear homogeneous differential polynomials of positive orders over $R$. Then the condition $\partial \operatorname{res}(f, g)=0$ is both necessary and sufficient for there to exist a common non-trivial solution of $f(x)=0$ and $g(x)=0$ in an extension of $K$.

We have also an analogue and slight reformulation of Theorem 3.10.
Theorem 3.14. Let $f(x), g(x) \in R_{L H}\{x\}$. Then $x \partial \operatorname{res}(f, g)$ is contained in the differential ideal $\langle f, g\rangle$.

From the above theorem we readily obtain an alternative proof that $\partial \operatorname{res}(f, g)=0$ is a necessary condition for the existence of a non-trivial common solution of $f(x)=0$ and $g(x)=0$. The details are left to the reader.

## References

[1] Carra'-Ferro, G. A resultant theory for systems of linear PDEs. In Proc. of Modern Group Analysis, 1994.
[2] Carra'-Ferro, G. A resultant theory for the systems of two ordinary algebraic differential equations. AAECC 8/6, 539-560, 1997.
[3] Chardin, M. Differential resultants and subresultants. In Proc. Fundamentals of Computation Theory 1991. In LNCS Vol. 529, Springer-Verlag, 1991.
[4] Collins, G.E. The calculation of multivariate polynomial resultants. J.ACM 18/4, 515-532, 1971.
[5] Kaplansky, I. An Introduction to Differential Algebra, Hermann, 1957.
[6] Kolchin, E.R. Algebraic matric groups and the Picard-Vessiot theory of homogeneous linear ODEs. Ann. of Math. 49, 1-42, 1948.
[7] Kolchin, E.R. Existence theorems connected with the Picard-Vessiot theory of homogeneous linear ODEs. Bull. Amer. Math. Soc. 54, 927-932, 1948.
[8] Ritt, J. F. Differential Equations from the Algebraic Standpoint. AMS Coll. Publ. Vol. 14, New York, 1932.
[9] Zwillinger, D. Handbook of Differential Equations Third Edn. Academic Press, 1998.

