taken from Lecture Notes "Commutative Algebra and Algebraic Geometry", Summer 2020

8. Rational parametrization of curves

Most of the results in this chapter are obvious for lines.
For this reason, and for simplicity in the explanation, we exclude lines from our treatment of rational parametrizations.
K is an algebraically closed field of characteristic 0.

8.1 Rational curves and parametrizations

Some plane algebraic curves can be expressed by means of rational parametrizations, i.e. pairs of univariate rational functions that, except for finitely many exceptions, represent all the points on the curve.

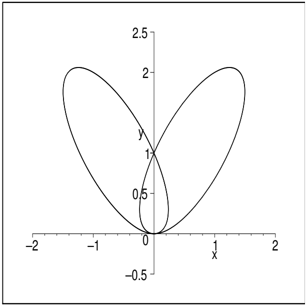
For instance, the parabola $y = x^2$ can also be described as the set $\{(t, t^2) \mid t \in K\}$; in this case, all affine points on the parabola are given by the parametrization (t, t^2).

Or compare Example 6.2.2. Also, the tacnode curve defined in $\mathbb{A}^2(\mathbb{C})$ by the polynomial

$$f(x,y) = 2x^4 - 3x^2y + y^2 - 2y^3 + y^4$$

can be represented, for instance, as

$$\left\{ \left(\frac{t^3 - 6t^2 + 9t - 2}{2t^4 - 16t^3 + 40t^2 - 32t + 9}, \quad \frac{t^2 - 4t + 4}{2t^4 - 16t^3 + 40t^2 - 32t + 9} \right) \;\middle|\; t \in \mathbb{C} \right\}$$



However, not all plane algebraic curves can be rationally parametrized, as we will see in Example 8.1.1.
In this section we introduce the notion of rational or parametrizable curve and we study the main properties and characterizations of this type of curves.
In the next sections we will show how to check the rationality by algorithmic methods and how to actually compute rational parametrizations of algebraic curves.

In Definition 6.2.5 we have introduced the notion of rationality for an arbitrary variety by means of rational isomorphisms. Now, we give a particular definition for the case of plane curves. Later, in Theorem 8.1.7, we prove that both definitions are equivalent.

Definition 8.1.1. The affine curve $\mathcal{C}$ in $\mathbb{A}^2(K)$ defined by the square–free polynomial $f(x, y)$ is rational (or parametrizable) if there are rational functions $\chi_1(t), \chi_2(t) \in K(t)$ such that

1. for almost all (i.e. for all but a finite number of exceptions) $t_0 \in K$, $(\chi_1(t_0), \chi_2(t_0))$ is a point on $\mathcal{C}$, and
2. for almost every point $(x_0, y_0) \in \mathcal{C}$ there is a $t_0 \in K$ such that $(x_0, y_0) = (\chi_1(t_0), \chi_2(t_0))$.

In this case $(\chi_1(t), \chi_2(t))$ is called a (rational affine) parametrization of $\mathcal{C}$.
We say that $(\chi_1(t), \chi_2(t))$ is in reduced form if the rational functions $\chi_1(t), \chi_2(t)$ are in reduced form; i.e. if for $i = 1, 2$ the gcd of the numerator and the denominator of χ_i is trivial.

Definition 8.1.2. The projective curve $\mathcal{C}$ in $\mathbb{P}^2(K)$ defined by the square–free homogeneous polynomial $F(x, y, z)$ is rational (or parametrizable) if there are polynomials $\chi_1(t), \chi_2(t), \chi_3(t) \in K[t]$, $\gcd(\chi_1, \chi_2, \chi_3) = 1$, such that

1. for almost all $t_0 \in K$, $(\chi_1(t_0) : \chi_2(t_0) : \chi_3(t_0))$ is a point on $\mathcal{C}$, and
2. for almost every point $(x_0 : y_0 : z_0) \in \mathcal{C}$ there is a $t_0 \in K$ such that $(x_0 : y_0 : z_0) = (\chi_1(t_0) : \chi_2(t_0) : \chi_3(t_0))$.

In this case, $(\chi_1(t), \chi_2(t), \chi_3(t))$ is called a (rational projective) parametrization of $\mathcal{C}$.

If $\mathcal{C}$ is an affine rational curve, and $\mathcal{P}(t)$ is a rational affine parametrization of $\mathcal{C}$ over K, we write its components either as

$$\mathcal{P}(t) = \left(\frac{\chi_{1\,1}(t)}{\chi_{1\,2}(t)}, \frac{\chi_{2\,1}(t)}{\chi_{2\,2}(t)} \right),$$

where $\chi_{i,j}(t) \in K[t]$, or as

$$\mathcal{P}(t) = (\chi_1(t), \chi_2(t)),$$

where $\chi_i(t) \in K(t)$. Similarly, rational projective parametrizations are expressed as

$$\mathcal{P}(t) = (\chi_1(t), \chi_2(t), \chi_3(t)),$$

where $\chi_i(t) \in K[t]$ and $\gcd(\chi_1, \chi_2, \chi_3) = 1$.

Furthermore, associated with a given parametrization $\mathcal{P}(t)$ we consider the polynomials

$$G_1^{\mathcal{P}}(s,t) = \chi_{1\,1}(s)\chi_{1\,2}(t) - \chi_{1\,2}(s)\chi_{1\,1}(t),$$

$$G_2^{\mathcal{P}}(s,t) = \chi_{2\,1}(s)\chi_{2\,2}(t) - \chi_{2\,2}(s)\chi_{2\,1}(t)$$

as well as the polynomials

$$H_1^{\mathcal{P}}(t,x,y) = x \cdot \chi_{1\,2}(t) - \chi_{1\,1}(t), \quad H_2^{\mathcal{P}}(t,x,y) = y \cdot \chi_{2\,2}(t) - \chi_{2\,1}(t).$$

The roots (s_0, t_0) of the polynomials $G_i^{\mathcal{P}}$ express that s_0 and t_0 generate the same curve point. The polynomials $H_i^{\mathcal{P}}$ play an important role in the implicitization of a parametrically given curve.

Remark.

(1) Later we will introduce the notion of local parametrization of a curve over K, not necessarily rational. Rational parametrizations are also called global parametrizations, and can only be achieved for genus zero curves (see Theorem 8.1.8.). On the other hand, since $K(t) \subset K((t))$, it is clear that any global parametrization is a local parametrization. By interpreting the numerator and denominator of a global parametrization as formal power series, and formally dividing, we get exactly a local parametrization.

(2) The notion of rational parametrization can be stated by means of rational maps as we did in Definition 6.2.5. More precisely, let $\mathcal{C}$ be a rational affine curve and $\mathcal{P}(t) \in K(t)^2$ a parametrization of $\mathcal{C}$. If $t_0 \in K$ is such that the denominators of the rational functions in $\mathcal{P}(t)$ are defined, then $\mathcal{P}(t_0) \in \mathcal{C}$. Thus, the parametrization $\mathcal{P}(t)$ induces the rational map

$$\begin{array}{rccc} \mathcal{P}: & \mathbb{A}^1(K) & \longrightarrow & \mathcal{C} \\ & t & \longmapsto & \mathcal{P}(t), \end{array}$$

and $\mathcal{P}(\mathbb{A}^1(K))$ is a dense (in the Zariski topology) subset of $\mathcal{C}$.

(3) Every rational parametrization $\mathcal{P}(t)$ defines a monomorphism from the field of rational functions $K(\mathcal{C})$ to $K(t)$ as follows (see proof of Theorem 8.1.6.):

$$\begin{array}{rccc} \varphi : & K(\mathcal{C}) & \longrightarrow & K(t) \\ & R(x,y) & \longmapsto & R(\mathcal{P}(t)). \end{array}$$

Example 8.1.1. An example of an irreducible curve which is not rational is the projective cubic $\mathcal{C}$, defined over $\mathbb{C}$, by $x^3 + y^3 = z^3$. Suppose that $\mathcal{C}$ is rational, and let $(\chi_1(t), \chi_2(t), \chi_3(t))$ be a parametrization of $\mathcal{C}$ in reduced form. Then

$$\chi_1^3 + \chi_2^3 - \chi_3^3 = 0.$$

Differentiating this equation by t we get

$$3 \cdot (\chi_1' \chi_1^2 + \chi_2' \chi_2^2 - \chi_3' \chi_3^2) = 0.$$

So $\chi_1^2, \chi_2^2, \chi_3^2$ are a solution of the system of homogeneous linear equations with coefficient matrix

$$\begin{pmatrix} \chi_1 & \chi_2 & -\chi_3 \\ \chi_1' & \chi_2' & -\chi_3' \end{pmatrix} .$$

By elementary line operations we reduce this coefficient matrix to [1]

$$\begin{pmatrix} \chi_2\chi_1' - \chi_2'\chi_1 & 0 & \chi_2'\chi_3 - \chi_2\chi_3' \\ 0 & \chi_2\chi_1' - \chi_2'\chi_1 & \chi_3'\chi_1 - \chi_3\chi_1' \end{pmatrix} .$$

So

$$\chi_1^2 : \chi_2^2 : \chi_3^2 = -\chi_2\chi_3' + \chi_3\chi_2' : -\chi_3\chi_1' + \chi_1\chi_3' : \chi_1\chi_2' - \chi_2\chi_1'.$$

Since χ_1, χ_2, χ_3 are relatively prime, this proportionality implies

$$\chi_1^2 \,|\, (\chi_2\chi_3' - \chi_3\chi_2'), \quad \chi_2^2 \,|\, (\chi_3\chi_1' - \chi_1\chi_3'), \quad \chi_3^2 \,|\, (\chi_1\chi_2' - \chi_2\chi_1').$$

Suppose $\deg(\chi_1) \geq \deg(\chi_2), \deg(\chi_3)$. Then the first divisibility implies $2\deg(\chi_1) \leq \deg(\chi_2) + \deg(\chi_3) - 1$, a contradiction.
Similarly we see that $\deg(\chi_2) \geq \deg(\chi_1), \deg(\chi_3)$ and $\deg(\chi_3) \geq \deg(\chi_1), \deg(\chi_2)$ are impossible.
Thus, there can be no parametrization of $\mathcal{C}$. $\square$

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[1] $-\chi_2' \cdot l1 + \chi_2 \cdot l2$ and $\chi_1' \cdot l1 - \chi_1 \cdot l2$

Theorem 8.1.1. *Any rational curve is irreducible.*

Proof: Let $\mathcal{C}$ be a rational affine curve (similarly if $\mathcal{C}$ is projective) parametrized by a rational parametrization $\mathcal{P}(t)$. First observe that the ideal of $\mathcal{C}$ consists of the polynomials vanishing at $\mathcal{P}(t)$, i.e.

$$I(\mathcal{C}) = \{h \in K[x, y] \mid h(\mathcal{P}(t)) = 0\}.$$

Indeed, if $h \in I(\mathcal{C})$ then $h(P) = 0$ for all $P \in \mathcal{C}$. In particular h vanishes on all points generated by the parametrization, and hence $h(\mathcal{P}(t)) = 0$. Conversely, let $h \in K[x, y]$ be such that $h(\mathcal{P}(t)) = 0$. Therefore, h vanishes on all points of the curve generated by $\mathcal{P}(t)$, i.e. on all points of $\mathcal{C}$ with finitely many exceptions. So, it vanishes on $\mathcal{C}$, i.e. $h \in I(\mathcal{C})$.

Finally, in order to prove that $\mathcal{C}$ is irreducible, we prove that $I(\mathcal{C})$ is prime. Let $h_1 \cdot h_2 \in I(\mathcal{C})$. Then $h_1(\mathcal{P}(t)) \cdot h_2(\mathcal{P}(t)) = 0$. Thus, either $h_1(\mathcal{P}(t)) = 0$ or $h_2(\mathcal{P}(t)) = 0$. Therefore, either $h_1 \in I(\mathcal{C})$ or $h_2 \in I(\mathcal{C})$. $\square$

Lemma 8.1.2. *Let $\mathcal{C}$ be an irreducible affine curve and $\mathcal{C}^*$ its corresponding projective curve. Then $\mathcal{C}$ is rational if and only if $\mathcal{C}^*$ is rational. Furthermore, a parametrization of $\mathcal{C}$ can be computed from a parametrization of $\mathcal{C}^*$ and vice versa.*

Proof: Let

$$(\chi_1(t), \chi_2(t), \chi_3(t))$$

be a parametrization of $\mathcal{C}^*$. Observe that $\chi_3(t) \neq 0$, since the curve $\mathcal{C}^*$ can have only finitely many points at infinity. Hence,

$$\left(\frac{\chi_1(t)}{\chi_3(t)}, \frac{\chi_2(t)}{\chi_3(t)}\right)$$

is a parametrization of the affine curve $\mathcal{C}$.

Conversely, a rational parametrization of $\mathcal{C}$ can always be extended to a parametrization of $\mathcal{C}^*$ by setting the z–coordinate to 1. $\square$

Lemma 8.1.3. *Let $\chi_1(t), \chi_2(t) \in K(t)$ be rational functions in reduced form, not both of them constant. Then,*

$$\mathcal{P}(t) = (\chi_1(t), \chi_2(t))$$

parametrizes an irreducible plane curve $\mathcal{C}$ over K. Moreover, if none of the two rational functions is constant and $f(x, y)$ is the defining polynomial of $\mathcal{C}$, there exists $r \in \mathbb{N}$ such that

$$\mathrm{res}_t(H_1^{\mathcal{P}}(t, x, y), H_2^{\mathcal{P}}(t, x, y)) = (f(x, y))^r.$$

Note that in the second part of the statement of this lemma, we require that the parametrization should not have a constant component. This is not a loss of generality since this situation corresponds to the lines $x = \lambda$ or $y = \lambda$, for some $\lambda \in K$.

Proof: If one of the two rational functions is constant, then $\mathcal{P}(t)$ parametrizes a horizontal or vertical line. Suppose $\mathcal{P}(t) = (\chi_1(t), a)$. Then $f(x, y) = y - a$, $H_1 = x \cdot \chi_{12}(t) - \chi_{11}(t)$, $H_2 = y - a$. So $\mathrm{res}_t(H_1, H_2) = (y - a)^{\deg(\chi_1)}$.

Now, let us assume that none of the components of $\mathcal{P}(t)$ is constant.
Let $\chi_i(t) = \frac{\chi_{i,1}(t)}{\chi_{i,2}(t)}$, and let

$$h(x, y) = \mathrm{res}_t(H_1^{\mathcal{P}}(t, x, y), H_2^{\mathcal{P}}(t, x, y)).$$

First we observe that $H_1^{\mathcal{P}}$ and $H_2^{\mathcal{P}}$ are irreducible, because $\chi_1(t)$ and $\chi_2(t)$ are in reduced form. Hence $H_1^{\mathcal{P}}$ and $H_2^{\mathcal{P}}$ do not have common factors. Therefore, $h(x, y)$ is not the zero polynomial. Furthermore, h cannot be a constant polynomial either. Indeed: let $t_0 \in K$ be such that $\chi_{1\,2}(t_0)\chi_{2\,2}(t_0) \neq 0$. Then $H_1^{\mathcal{P}}(t_0, \mathcal{P}(t_0)) = H_2^{\mathcal{P}}(t_0, \mathcal{P}(t_0)) = 0$. So $h(\mathcal{P}(t_0)) = 0$, and since h is not the zero polynomial it cannot be constant.

Now, we consider the square–free part $h'(x, y)$ of $h(x, y)$ and the plane curve $\mathcal{C}$ defined by $h'(x, y)$ over K. Let us see that $\mathcal{P}(t)$ parametrizes $\mathcal{C}$. For this purpose, we check the conditions introduced in Definition 8.1.1.

1. Let $t_0 \in K$ be such that $\chi_{1\,2}(t_0)\chi_{2\,2}(t_0) \neq 0$. Reasoning as above, we see that $h(\mathcal{P}(t_0)) = 0$. So $h'(\mathcal{P}(t_0)) = 0$, and hence $\mathcal{P}(t_0)$ is on $\mathcal{C}$.
2. Let c_1, c_2 be the leading coefficients of $H_1^{\mathcal{P}}, H_2^{\mathcal{P}}$ w.r.t. t, respectively. Note that $c_1 \in K[x], c_2 \in K[y]$ are of degree at most 1. For every (x_0, y_0) on $\mathcal{C}$ such that $c_1(x_0) \neq 0$ or $c_2(y_0) \neq 0$ (note that there is at most one point in K^2 where c_1 and c_2 vanish simultaneously), we have $h(x_0, y_0) = 0$. Thus, since h is a resultant, there exists $t_0 \in K$ such that $H_1^{\mathcal{P}}(t_0, x_0, y_0) = H_2^{\mathcal{P}}(t_0, x_0, y_0) = 0$. Also, observe that $\chi_{1\,2}(t_0) \neq 0$ since otherwise the first component of the parametrization would not be in reduced form. Similarly, $\chi_{2\,2}(t_0) \neq 0$. Thus, $(x_0, y_0) = \mathcal{P}(t_0)$. Therefore, almost all points on $\mathcal{C}$ are generated by $\mathcal{P}(t)$.

Now by Theorem 8.1.1. it follows that h' is irreducible. Therefore, there exists $r \in \mathbb{N}$ such that $h(x, y) = (h'(x, y))^r$.

Theorem 8.1.4. *An irreducible curve $\mathcal{C}$, defined by $f(x, y)$, is rational if and only if there exist rational functions $\chi_1(t), \chi_2(t) \in K(t)$, not both constant, such that $f(\chi_1(t), \chi_2(t)) = 0$. In this case, $(\chi_1(t), \chi_2(t))$ is a rational parametrization of $\mathcal{C}$.*

Proof: Let $\mathcal{C}$ be rational. So there exist rational functions $\chi_1, \chi_2 \in K(t)$ satisfying conditions (1) and (2) in Definition 8.1.1. Obviously not both rational functions χ_i are constant, and clearly $f(\chi_1(t), \chi_2(t)) = 0$.

Conversely, let $\chi_1, \chi_2 \in K(t)$, not both constant, be such that $f(\chi_1(t), \chi_2(t))$ is identically zero. Let $\mathcal{D}$ be the irreducible plane curve defined by $(\chi_1(t), \chi_2(t))$ (see Lemma 8.1.3). Then $\mathcal{C}$ and $\mathcal{D}$ are both irreducible, because of Theorem 8.1.1, and have infinitely many points in common. Thus, by Bézout's theorem one concludes that $\mathcal{C} = \mathcal{D}$. Hence, $(\chi_1(t), \chi_2(t))$ is a parametrization of $\mathcal{C}$. $\square$

An alternative characterization of rationality in terms of field theory is given in Theorem 8.1.6. This theorem can be seen as the geometric version of Lüroth's Theorem. Lüroth's Theorem appears in basic text books on algebra such as [Wae70]. Here we do not give a proof of this result.

Theorem 8.1.5. (Lüroth's Theorem) *Let $\mathbb{L}$ be a field (not necessarily algebraically closed). Then every subfield $\mathbb{K}$ of $\mathbb{L}(t)$, where t is a transcendental element over $\mathbb{L}$, such that $\mathbb{K}$ strictly contains $\mathbb{L}$, is $\mathbb{L}$-isomorphic to $\mathbb{L}(t)$.*

Theorem 8.1.6. *An irreducible affine curve $\mathcal{C}$ is rational if and only if the field of rational functions on $\mathcal{C}$, i.e. $K(\mathcal{C})$, is isomorphic to $K(t)$ (t a transcendental element).*

Proof: Let $f(x,y)$ be the defining polynomial of $\mathcal{C}$, and let $\mathcal{P}(t)$ be a parametrization of $\mathcal{C}$. We consider the map

$$\begin{array}{cccc} \varphi_{\mathcal{P}}: & K(\mathcal{C}) & \longrightarrow & K(t) \\ & R(x,y) & \longmapsto & R(\mathcal{P}(t)). \end{array}$$

First we observe that $\varphi_{\mathcal{P}}$ is well-defined. Let $\frac{p_1}{q_1}, \frac{p_2}{q_2}$, where $p_i, q_i \in K[x,y]$, be two different expressions of the same element in $K(\mathcal{C})$. Then f divides $p_1q_2 - q_1p_2$. By Theorem 8.1.4, $f(\mathcal{P}(t))$ is identically equal to zero, and therefore $p_1(\mathcal{P}(t))q_2(\mathcal{P}(t)) - q_1(\mathcal{P}(t))p_2(\mathcal{P}(t))$ is also identically zero. Furthermore, since $q_1 \neq 0$ in $K(\mathcal{C})$, we have $q_1(\mathcal{P}(t)) \neq 0$. Similarly $q_2(\mathcal{P}(t)) \neq 0$. Therefore, $\varphi_{\mathcal{P}}(\frac{p_1}{q_1}) = \varphi_{\mathcal{P}}(\frac{p_2}{q_2})$. Now, since $\varphi_{\mathcal{P}}$ is not the zero homomorphism, and $\varphi_{\mathcal{P}}$ is injective [2] one has that $\varphi_{\mathcal{P}}$ defines an isomorphism of $K(\mathcal{C})$ onto a subfield of $K(t)$ that properly contains K. Thus, by Lüroth's Theorem, this subfield, and $K(\mathcal{C})$ itself, must be isomorphic to $K(t)$.

[2] $\frac{p_1}{q_1}(\mathcal{P}) = \frac{p_2}{q_2}(\mathcal{P})$ implies $p_1q_2 - p_2q_1 = 0$ on infinitely many points, so it is identically 0

Conversely, let $\psi : K(\mathcal{C}) \to K(t)$ be an isomorphism and $\chi_1(t) = \psi(x), \chi_2(t) = \psi(y)$. Clearly, since the image of ψ is $K(t)$, χ_1 and χ_2 cannot both be constant. Furthermore

$$f(\chi_1(t), \chi_2(t)) = f(\psi(x), \psi(y)) = \psi(f(x, y)) = 0.$$

Hence, by Theorem 8.1.4, $(\chi_1(t), \chi_2(t))$ is a rational parametrization of $\mathcal{C}$. $\square$

Rationality can also be established by means of rational maps. The next characterization shows that Definitions 8.1.1 and 6.2.5 (for plane curves) are equivalent. Furthermore, it implies that the notions of rationality and unirationality are equivalent for plane curves.

Theorem 8.1.7. *An affine algebraic curve $\mathcal{C}$ is rational if and only if it is birationally equivalent to K (i.e. the affine line $\mathbb{A}^1(K)$).*

Proof: By Theorem 6.2.3. one has that $\mathcal{C}$ is birationally equivalent to K if and only if $K(\mathcal{C})$ is isomorphic to $K(t)$. Thus, by Theorem 8.1.6 we get the desired result. $\square$

The following theorem states that rational curves are precisely those with genus zero. In fact, all irreducible conics are rational, and an irreducible cubic is rational if and only if it has a double point. We get this theorem by using the fact (which we have not proved) that the genus is invariant under birational maps.

Theorem 8.1.8. *An algebraic curve $\mathcal{C}$ is rational if and only if* $\mathrm{genus}(\mathcal{C}) = 0$.

8.2 Proper parametrizations

Although the implicit representation for a plane curve is unique, up to a constant, there exist infinitely many different parametrizations of the same rational curve. For instance, for every $i \in \mathbb{N}$, (t^i, t^{2i}) parametrizes the parabola $y = x^2$. Obviously (t, t^2) is the parametrization of lowest degree in this family. Such parametrizations are called proper parametrizations.

The parametrization algorithms presented in this chapter always output proper parametrizations. Furthermore, there are algorithms for determining whether a given parametrization of a plane curve is proper, and if that is not the case, for transforming it to a proper one. In Section 6.1 we will describe these methods.

In this section, we introduce the notion of proper parametrization and we study some of the main properties. For this purpose, in the following we assume that $\mathcal{C}$ is an affine rational plane curve, and $\mathcal{P}(t)$ is a rational affine parametrization of $\mathcal{C}$.

Definition 8.2.1. An affine parametrization $\mathcal{P}(t)$ of a rational curve $\mathcal{C}$ is proper if the map

$$\begin{array}{rccc} \mathcal{P}: & \mathbb{A}^1(K) & \longrightarrow & \mathcal{C} \\ & t & \longmapsto & \mathcal{P}(t) \end{array}$$

is birational, or equivalently, if almost every point on $\mathcal{C}$ is generated by exactly one value of the parameter t.
We define the inversion of a proper parametrization $\mathcal{P}(t)$ as the inverse rational mapping of $\mathcal{P}$, and we denote it by $\mathcal{P}^{-1}$.

Lemma 8.2.1. *Every rational curve can be properly parametrized.*

Proof: From Theorem 8.1.7. one deduces that any rational curve $\mathcal{C}$ is birationally equivalent to $\mathbb{A}^1(K)$. Therefore, any rational curve can be properly parametrized. $\square$

The notion of properness can also be stated algebraically in terms of fields of rational functions. From Theorem 6.2.3 we deduce that a rational parametrization $\mathcal{P}(t)$ is proper if and only if the induced monomorphism $\varphi_{\mathcal{P}}$ (see Remark to Theorem 8.1.6)

$$\begin{array}{rccc} \varphi_{\mathcal{P}}: & K(\mathcal{C}) & \longrightarrow & K(t) \\ & R(x,y) & \longmapsto & R(\mathcal{P}(t)). \end{array}$$

is an isomorphism. Therefore, $\mathcal{P}(t)$ is proper if and only if the mapping $\varphi_{\mathcal{P}}$ is surjective, that is, if and only if $\varphi_{\mathcal{P}}(K(\mathcal{C})) = K(\mathcal{P}(t)) = K(t)$. More precisely, we have the following theorem.

Theorem 8.2.2. *Let $\mathcal{P}(t)$ be a rational parametrization of a plane curve $\mathcal{C}$. Then, the following statements are equivalent:*

(1) *$\mathcal{P}(t)$ is proper.*

(2) *The monomorphism $\varphi_{\mathcal{P}}$ induced by $\mathcal{P}$ is an isomorphism.*

(3) *$K(\mathcal{P}(t)) = K(t)$.*

Remark. We have introduced the notion of properness for affine parametrizations. For projective parametrizations the notion can be extended by asking the rational map, obtained by homogenizing the projective parametrization, from $\mathbb{P}^1(K)$ onto the curve to be birational. Moreover, if $\mathcal{C}$ is an irreducible affine curve and $\mathcal{C}^\star$ is its projective closure, then $K(\mathcal{C}) = K(\mathcal{C}^\star)$. Thus, taking into account Theorem 8.2.2. one has that the properness of affine and projective parametrizations are equivalent.

Now, we characterize proper parametrizations by means of the degree of the corresponding rational curve. To state this result, we first introduce the notion of degree of a parametrization.

Definition 8.2.2. Let $\chi(t) \in K(t)$ be a non-zero rational function in reduced form. If $\chi(t)$ is not zero, the degree of $\chi(t)$ is the maximum of the degrees of the numerator and denominator of $\chi(t)$. If $\chi(t)$ is zero, we define its degree to be -1. We denote the degree of $\chi(t)$ as $\deg(\chi(t))$.
Rational functions of degree 1 are called linear. □

Obviously the degree is multiplicative with respect to the composition of rational functions. Furthermore, invertible rational functions are exactly the linear rational functions.

Definition 8.2.3. We define the degree of a rational affine parametrization $\mathcal{P}(t) = (\chi_1(t), \chi_2(t))$ as the maximum of the degrees of its rational components; i.e.

$$\deg(\mathcal{P}(t)) = \max\{\deg(\chi_1(t)), \deg(\chi_2(t))\} \ .$$

Lemma 8.2.3. *Let $\mathcal{P}(t)$ be a proper parametrization of a rational affine plane curve $\mathcal{C}$, and let $\mathcal{P}'(t)$ be any other rational parametrization of $\mathcal{C}$. Then*

(1) *there exists a rational function $R(t) \in K(t) \setminus K$ such that $\mathcal{P}'(t) = \mathcal{P}(R(t))$;*

(2) *$\mathcal{P}'(t)$ is proper if and only if there exists a linear rational function $L(t) \in K(t)$ such that $\mathcal{P}'(t) = \mathcal{P}(L(t))$.*

Proof: (1) We consider the following diagram

$$\begin{tikzcd}
\mathbb{A}^1(K) \arrow[r, "\mathcal{P}"] & \mathcal{C} \subset \mathbb{A}^2(K) \\
& \mathbb{A}^1(K) \arrow[ul, "\mathcal{P}^{-1} \circ \mathcal{P}'"] \arrow[u, "\mathcal{P}'"']
\end{tikzcd}$$

Then, since $\mathcal{P}$ is a birational mapping, it is clear that $R(t) = \mathcal{P}^{-1}(\mathcal{P}'(t)) \in K(t)$.

(2) If $\mathcal{P}'(t)$ is proper, then from the diagram above we see that $\varphi = \mathcal{P}^{-1} \circ \mathcal{P}'$ is a birational mapping from $\mathbb{A}^1(K)$ onto $\mathbb{A}^1(K)$. Hence, by Theorem 6.2.3 one has that φ induces an automorphism $\tilde{\varphi}$ of $K(t)$ defined as:

$$\begin{array}{rccc} \tilde{\varphi}: & K(t) & \longrightarrow & K(t) \\ & t & \longmapsto & \varphi(t). \end{array}$$

Therefore, since K-automorphisms of $K(t)$ are the invertible rational functions (see e.g. [Wae70]), we see that $\tilde{\varphi}$ is our linear rational function.

Conversely, let ψ be the birational mapping from $\mathbb{A}^1(K)$ onto $\mathbb{A}^1(K)$ defined by the linear rational function $L(t) \in K(t)$. Then, it is clear that $\mathcal{P}' = \mathcal{P} \circ \psi : \mathbb{A}^1(K) \to \mathcal{C}$ is a birational mapping, and therefore $\mathcal{P}'(t)$ is proper. $\square$

The proofs of the following statements are technical, and we omit them here. But they are given in the Lecture Notes.

Theorem 8.2.5. *Let $\mathcal{C}$ be a rational affine curve defined over K with defining polynomial $f(x,y) \in K[x,y]$, and let $\mathcal{P}(t) = (\chi_1(t), \chi_2(t))$ be a parametrization of $\mathcal{C}$. Then $\mathcal{P}(t)$ is proper if and only if*

$$\deg(\mathcal{P}(t)) = \max\{\deg_x(f), \deg_y(f)\}.$$

Furthermore, if $\mathcal{P}(t)$ is proper, then $\deg(\chi_1(t)) = \deg_y(f)$, and $\deg(\chi_2(t)) = \deg_x(f)$.

The next corollary follows from Theorem 8.2.5 and Lemma 8.2.3.

Corollary. *Let $\mathcal{C}$ be a rational affine plane curve defined by $f(x,y) \in K[x,y]$. Then the degree of any rational parametrization of $\mathcal{C}$ is a multiple of $\max\{\deg_x(f), \deg_y(f)\}$.*

Example 8.2.1. We consider the rational quintic $\mathcal{C}$ defined by the polynomial $f(x,y) = y^5 + x^2y^3 - 3\,x^2y^2 + 3\,x^2y - x^2$. Theorem 8.2.5 ensures that any rational proper parametrization of $\mathcal{C}$ must have a first component of degree 5, and a second component of degree 2. It is easy to check that

$$\mathcal{P}(t) = \left(\frac{t^5}{t^2+1}, \frac{t^2}{t^2+1}\right)$$

parametrizes properly $\mathcal{C}$. Note that $f(\mathcal{P}(t)) = 0$.

8.3 Parametrization by Lines

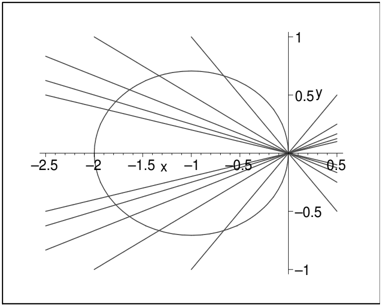


Figure 8.2: Ellipse $x^2 + 2x + 2y^2 = 0$ and pencil $\mathcal{H}(t)$

We start with the simple case of irreducible conics.

$$f(x,y) = f_2(x,y) + f_1(x,y) + f_0(x,y)$$

assume w.l.o.g. that $\mathcal{C}$ passes through the origin, so $f_0(x,y) = 0$. Let $\mathcal{H}(t)$ be the linear system of lines through the origin, the elements of $\mathcal{H}(t)$ being parametrized by their slope t. So the defining polynomial of $\mathcal{H}(t)$ is $h(x,y,t) = y - tx$.

intersection points of a generic element of $\mathcal{H}(t)$ and $\mathcal{C}$:

$$\begin{cases} y = tx \\ f(x, y) = 0 \end{cases}$$

Solve in the variables x, y. We get $f_2(x, tx) = -f_1(x, tx)$, and further $x^2 \cdot f_2(1, t) = -x \cdot f_1(1, t)$.
So the solution points are

$$P = (0, 0) \quad \text{and} \quad Q = \left(-\frac{f_1(1, t)}{f_2(1, t)}, -\frac{t \cdot f_1(1, t)}{f_2(1, t)}\right).$$

Note that $f_1(x, y)$ is not identically zero, since $\mathcal{C}$ is an irreducible curve. Therefore, Q depends on the parameter t. Furthermore, the affine point Q is not reachable by at most two particular values of t, namely the roots of the quadratic form $f_2(x, y)$. Thus, for all but finitely many values of $t \in K$, $\mathcal{H}(t)$ and $\mathcal{C}$ intersects exactly at two different affine points (see Figure 8.2). The intersection point Q depends rationally on the parameter of t of $\mathcal{H}(t)$, and it yields the desired parametrization of the conic.
So we have proved the following theorem.

Theorem 8.3.1. *The irreducible projective conic $\mathcal{C}$ defined by the polynomial $F(x, y, z) = f_2(x, y) + f_1(x, y)z$ (f_i a form of degree i, resp.), has the rational parametrization*

$$\mathcal{P}(t) = (-f_1(1, t), -tf_1(1, t), f_2(1, t)).$$

In this situation, using the previous theorem and making a suitable change of coordinates, one may derive the following parametrization algorithm for conics.

Algorithm CONIC-PARAMETRIZATION.
Given the defining polynomial $F(x, y, z)$ of an irreducible projective conic $\mathcal{C}$, the algorithm computes a rational parametrization.

1. Compute the homogeneous components f_2, f_1, f_0 of $F(x, y, 1)$.
2. If $(0 : 0 : 1) \in \mathcal{C}$ then return
$\mathcal{P}(t) = (-f_1(1, t) : -tf_1(1, t) : f_2(1, t))$.
3. Compute a point $(a : b : 1) \in \mathcal{C}$.
4. $g(x, y) = F(x + a, y + b, 1)$. Let $g_2(x, y)$ and $g_1(x, y)$ be the homogeneous components of $g(x, y)$ of degree 2 and 1, respectively.
5. Return
$\mathcal{P}(t) = (-g_1(1, t) + ag_2(1, t) : -tg_1(1, t) + bg_2(1, t) : g_2(1, t))$.

Remark. Note that, because of the geometric construction, the output parametrization of algorithm CONIC-PARAMETRIZATION is proper.
Moreover, if $\mathcal{P}_{\star,z}(t)$ is the affine parametrization of $\mathcal{C}_{\star,z}$ derived from $\mathcal{P}(t)$, then its inverse can be expressed as

$$\mathcal{P}_{\star,z}^{-1}(x,y) = \frac{y-b}{x-a}.$$

Similary for $\mathcal{C}_{\star,y}$ and $\mathcal{C}_{\star,z}$

Example 8.3.1.: Let $\mathcal{C}$ be the ellipse defined by
$f(x, y) = x^2 + 2y^2 - z^2$.
$(0 : 0 : 1)$ is not on $\mathcal{C}$.
We take a point on $\mathcal{C}$, for instance $(1 : 0 : 1)$ (Step 3).
Step 4 yields $g(x, y) = x^2 + 2x + y^2$ (see Figure 8.2).
Then, a parametrization of $\mathcal{C}$ is

$$\mathcal{P}(t) = (-1 + 2\,t^2 : -2\,t : 1 + 2\,t^2)\ .$$

In the affine plane $(z = 1)$ the corresponding parametrization of the ellipse is

$$\mathcal{P}(t) = \left(\frac{-1 + 2\,t^2}{1 + 2\,t^2}, \frac{-2\,t}{1 + 2\,t^2}\right)\ .$$

The rational inverse of the parametrization is then

$$\mathcal{P}^{-1}(x, y) = \frac{y}{x - 1}.$$

And indeed,

$$\mathcal{P}^{-1}(\mathcal{P}(t)) = \frac{(-2t)/(1 + 2t^2)}{(-1 + 2t^2)/(1 + 2t^2)\ - 1} = \frac{-2t}{-2} = t\ .$$

Obviously, this approach can be immediately generalized to the situation where we have an irreducible projective curve $\mathcal{C}$ of degree d with a $(d-1)$–fold point P. W.l.o.g. we consider that $P = (0:0:1)$, so the defining polynomial of $\mathcal{C}$ is of the form

$$F(x,y,z) = f_d(x,y) + f_{d-1}(x,y)z$$

(f_i a form of degree i, resp.). Of course, there can be no other singularity of $\mathcal{C}$, since otherwise the line passing through the two singularities would intersect $\mathcal{C}$ more than d times.

As above, we consider the linear system of lines $\mathcal{H}(t)$ through $(0:0:1)$. Intersecting $\mathcal{C}$ with an element of $\mathcal{H}$ we get the origin as an intersection point of multiplicity at least $d-1$. Reasoning as above, one has that since $\mathcal{C}$ is irreducible for all but finitely many values of t, P is an intersection point of multiplicity at most $d-1$. Thus, by Bézout's Theorem, we must get exactly one more intersection point Q depending rationally on the value of t. So the coordinates of Q are polynomials in t, in fact

$$Q = (-f_{d-1}(1,t) : -t \cdot f_{d-1}(1,t) : f_d(1,t)).$$

This is a rational parametrization of the curve $\mathcal{C}$.

Theorem 8.3.2. *Let $\mathcal{C}$ be an irreducible projective curve of degree d defined by the polynomial $F(x, y, z) = f_d(x, y) + f_{d-1}(x, y)z$ (f_i a form of degree i, resp.), i.e. having a $(d-1)$–fold point at $(0:0:1)$. Then $\mathcal{C}$ is rational and a rational parametrization is*

$$\mathcal{P}(t) = (-f_{d-1}(1, t), -tf_{d-1}(1, t), f_d(1, t)).$$

Applying the previous theorems, one may derive an algorithm for parametrizing by lines. For this purpose, one just has to move the base point of the pencil of lines to the origin. More precisely, one has the following algorithm.

Algorithm PARAMETRIZATION-BY-LINES.
Given the defining polynomial $F(x, y, z)$ of an irreducible projective curve $\mathcal{C}$ of degree $d > 1$, having a $(d-1)$–fold point, the algorithm computes a rational parametrization of $\mathcal{C}$.

1. Compute the $(d-1)$–fold point P of $\mathcal{C}$; if $d = 2$, take any point P on $\mathcal{C}$. W.l.o.g., perhaps after renaming the variables, let $P = (a : b : 1)$.
2. $g(x, y) := F(x + a, y + b, 1)$. Let $g_d(x, y)$ and $g_{d-1}(x, y)$ be the homogeneous components of $g(x, y)$ of degree d and $d - 1$, respectively.
3. Return $\mathcal{P}(t) = (-g_{d-1}(1, t) + a g_d(1, t) \; : \; -t g_{d-1}(1, t) + b g_d(1, t) \; : \; g_d(1, t))$.

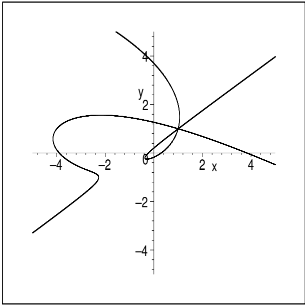
Remark. Note that, because of the underlying geometric construction, the parametrization computed by algorithm PARAMETRIZATION-BY-LINES is proper. Furthermore, if $\mathcal{P}_{\star,z}(t)$ is the affine parametrization of $\mathcal{C}_{\star,z}$ derived from $\mathcal{P}(t)$, then its inverse can be computed as follows. W.l.o.g., perhaps after renaming the variables, let $P = (a : b : 1)$ be the singularity of the curve. Then

$$\mathcal{P}_{\star,z}^{-1}(x, y) = \frac{y - b}{x - a}$$

is the inverse of $\mathcal{P}$.

Example 8.3.2.: Let $\mathcal{C}$ be the affine quartic curve defined by

$$\begin{aligned} f(x,y) = \quad & 1 + x - 15\,x^2 - 29\,y^2 + 30\,y^3 - 25\,xy^2 + x^3y \\ & +35\,xy + x^4 - 6\,y^4 + 6\,x^2y \ . \end{aligned}$$



$\mathcal{C}$ has an affine triple point at $(1,1)$.
In Step 2 we compute the polynomial

$$g(x,y) = 5\,x^3 + 6\,y^3 - 25\,xy^2 + x^3y + x^4 - 6\,y^4 + 9\,x^2y\ .$$

From the homogeneous forms of $g(x,y)$, in Step 3 we get the rational parametrization of $\mathcal{C}$

$$\mathcal{P}(t) = \left(\frac{4+6\,t^3-25\,t^2+8\,t+6\,t^4}{-1+6\,t^4-t}\,,\ \frac{4\,t+12\,t^4-25\,t^3+9\,t^2-1}{-1+6\,t^4-t}\right).$$

Furthermore, the inverse of the parametrization is given by

$$\mathcal{P}^{-1}(x,y) = \frac{y-1}{x-1}.$$

Example 8.3.3.: Let $\mathcal{C}$ be the affine quintic curve defined by

$$f(x,y) = \begin{array}{l} -\frac{75}{8}x^2y^2 + \frac{125}{8}x^3y - \frac{1875}{256}x^4 + x + y^4 + \frac{625}{16}x^3y^2 \\ -\frac{9375}{256}x^4y - \frac{125}{8}x^2y^3 + \frac{3125}{256}x^5 + y^5. \end{array}$$

$\mathcal{C}$ has a quadruple point at $(4:5:0)$.
In Step 4.1. we compute the polynomial

$$g(x,y) = 6400y^4 + 1024x^4 + 256y^5 + 256xy^4 + 5120xy^3$$

And, determining the homogeneous forms of $g(x,y)$, in Step 4.1.2., we get the rational parametrization of $\mathcal{C}$

$$\mathcal{P}(t) = \left(-4\,\frac{t^4\,(t+1)}{25\,t^4 + 20\,t^3 + 4}, \frac{t\left(20\,t^4 + 15\,t^3 + 4\right)}{25\,t^4 + 20\,t^3 + 4}\right).$$

Furthermore, taking into account the remark to the algorithm we have that

$$\mathcal{P}^{-1}(x,y) = y - \frac{5}{4}x\ .$$

Definition 8.3.1. The irreducible projective curve $\mathcal{C}$ is *parametrizable by lines* if there exists a linear system of curves $\mathcal{H}$ of degree 1 (i.e. a pencil of lines) such that

(1) $\dim(\mathcal{H}) = 1$,

(2) the intersection of a generic element in $\mathcal{H}$ and $\mathcal{C}$ contains a non–constant point whose coordinates depend rationally on the free parameter of $\mathcal{H}$.

We say that an irreducible affine curve is *parametrizable by lines* if its projective closure is parametrizable by lines. □

Theorem 8.3.3. *Let $\mathcal{C}$ be an irreducible projective plane curve of degree $d > 1$. The following statements are equivalent:*

(1) *$\mathcal{C}$ is parametrizable by a pencil of lines $\mathcal{H}(t)$.*

(2) *$\mathcal{C}$ has a point of multiplicity $d-1$ which is the base point of $\mathcal{H}(t)$.*

8.4 Parametrization by Adjoint Curves

Definition 8.4.1. We say that a linear system of curves $\mathcal{H}$ *parametrizes* $\mathcal{C}$ if it holds that

(1) $\dim(\mathcal{H}) = 1$,

(2) the intersection of a generic element in $\mathcal{H}$ and $\mathcal{C}$ contains a non–constant point whose coordinates depend rationally on the free parameter in $\mathcal{H}$,

(3) $\mathcal{C}$ is not a common component of any curves in $\mathcal{H}$.

Lemma 8.4.1. *Let $\mathcal{H}(t)$ be a linear system of curves parametrizing $\mathcal{C}$, then there exists only one non–constant intersection point $\mathcal{P}(t)$ of $\mathcal{H}(t)$ and $\mathcal{C}$ depending on t, and it is a proper parametrization of $\mathcal{C}$.*

Theorem 8.4.2. *Let $F(x, y, z)$ be the defining polynomial of $\mathcal{C}$, and let $H(t, x, y, z)$ be the defining polynomial of a linear system $\mathcal{H}(t)$ parametrizing $\mathcal{C}$. Then, the proper parametrization $\mathcal{P}(t)$ generated by $\mathcal{H}(t)$ is the solution in $\mathbb{P}^2(K(t))$ of the system of algebraic equations*

$$\left.\begin{array}{l} \mathrm{pp}_t(\mathrm{res}_y(F, H)) = 0 \\ \mathrm{pp}_t(\mathrm{res}_x(F, H)) = 0 \end{array}\right\}.$$

Definition 8.4.2. We say that a projective curve $\mathcal{C}'$ is an adjoint curve of the irreducible $\mathcal{C}$ if and only if the following holds:

(1) if P is a singular point of $\mathcal{C}$, then $\mathrm{mult}_P(\mathcal{C}') \geq \mathrm{mult}_P(\mathcal{C}) - 1$,

(2) if P is a neighboring singular point of $\mathcal{C}$, then $\mathrm{mult}_P(\mathcal{C}') \geq \mathrm{mult}_P(\mathcal{C}) - 1$.

We say that $\mathcal{C}'$ is an adjoint curve of degree k of $\mathcal{C}$, if $\mathcal{C}'$ is an adjoint of $\mathcal{C}$ and $\deg(\mathcal{C}') = k$. $\square$

All algebraic conditions required in the definition of adjoint curve are linear. Therefore if one fixes the degree, the set of all adjoint curves of $\mathcal{C}$ is a linear system of curves. In fact, if $\mathcal{C}$ has only ordinary singularities, then the set of adjoint curves of degree k of $\mathcal{C}$ is the linear system generated by the effective divisor

$$\sum_{P \in \mathrm{Sing}(\mathcal{C})} (\mathrm{mult}_P(\mathcal{C}) - 1)P.$$

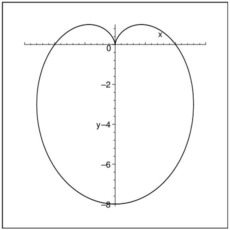
Definition 8.4.3. The set of all adjoints of $\mathcal{C}$ of degree k, $k \in \mathbb{N}$, is called the system of adjoints of $\mathcal{C}$ of degree k. We denote this system by $\mathcal{A}_k(\mathcal{C})$. $\square$

Theorem 8.4.6. *Let $\mathcal{S} \subset \mathcal{C} \setminus \operatorname{Sing}(\mathcal{C})$ be such that $\#(\mathcal{S}) = d - 3$. Then $\mathcal{A}_{d-2}(\mathcal{C}) \cap \mathcal{H}(d-2, \sum_{P \in \mathcal{S}} P)$ parametrizes $\mathcal{C}$.*

Algorithm PARAMETRIZATION-BY-ADJOINTS.
Given the defining polynomial $F(x, y, z)$ of a rational irreducible projective curve $\mathcal{C}$ of degree d the algorithm computes a rational parametrization of $\mathcal{C}$.

1. If $d \leq 3$ or $\operatorname{Sing}(\mathcal{C})$ contains only one point of multiplicity $d-1$ apply algorithm PARAMETRIZATION−BY−LINES.
2. Compute the defining polynomial of $\mathcal{A}_{d-2}(\mathcal{C})$.
3. Choose a set $\mathcal{S} \subset (\mathcal{C} \setminus \operatorname{Sing}(\mathcal{C}))$ such that $\#(\mathcal{S}) = d - 3$.
4. Compute the defining polynomial H of $\mathcal{H} = \mathcal{A}_{d-1}(\mathcal{C}) \cap \mathcal{H}(d-2, \sum_{P \in \mathcal{S}} P)$.
5. Return the solution in $\mathbb{P}^2(K(t))$ of $\{\mathrm{pp}_t(\mathrm{res}_y(F, H)) = 0, \mathrm{pp}_t(\mathrm{res}_x(F, H)) = 0\}$.

Example 8.4.1: Cardioid $\mathcal{C}$:
$f(x,y) = (x^2 + 4y + y^2)^2 - 16(x^2 + y^2)$.



$\mathcal{C}$ has a double point at the origin $(0,0)$ as the only affine singularity. But if we move to the associated projective curve $\mathcal{C}^*$ defined by the homogeneous polynomial

$$F(x,y,z) = (x^2 + 4yz + y^2)^2 - 16(x^2 + y^2)z^2,$$

we see that the singularities of $\mathcal{C}^*$ are

$$O = (0:0:1), \quad P_{1,2} = (1:\pm i:0).$$

$P_{1,2}$ is a family of conjugate algebraic points on $\mathcal{C}^*$. All of these singularities have multiplicity 2, so the genus of $\mathcal{C}^*$ is 0, i.e. it can be parametrized. So also the affine curve $\mathcal{C}$ is parametrizable.

In order to achieve a parametrization, we need a simple point on $\mathcal{C}^*$. Intersecting $\mathcal{C}^*$ by the line $x = 0$, we get of course the origin as a multiple intersection point. The other intersection point is

$$Q = (0 : -8 : 1).$$

So now we construct the system $\mathcal{H}$ of curves of degree 2, having $O, P_{1,2}$ and Q as base points of multiplicity 1. The full system of curves of degree 2 is of the form

$$a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz$$

for arbitrary coefficients $a_1, \ldots, a_6$.

Requiring that O be a base point leads to the linear equation

$$a_3 = 0.$$

Requiring that $P_{1,2}$ should be base points of $\mathcal{H}$ leads to the equations

$$a_4 = 0, \quad a_1 - a_2 = 0 \ .$$

Finally, to make Q a base point we have to satisfy

$$64a_2 + a_3 - 8a_6 = 0.$$

This leaves exactly 2 parameters unspecified, say a_1 and a_5. Since curves are defined uniquely by polynomials only up to a nonzero constant factor, we can set one of these parameters to 1. Thus, the system $\mathcal{H}$ depends on 1 free parameter $a_1 = t$, and its defining equation is

$$H(x, y, z, t) = tx^2 + ty^2 + xz + 8tyz.$$

The affine version is defined by

$$h(x, y, t) = tx^2 + ty^2 + x + 8ty.$$

Now we determine the free intersection point of $\mathcal{H}$ and $\mathcal{C}$. The non–constant factors of $\mathrm{res}_x(f(x,y), h(x,y,t))$ are

$$\begin{aligned} & y^2, \\ & y+8, \\ & (256t^4+32t^2+1)y+(2048t^4-128t^2). \end{aligned}$$

The first two factors correspond to the affine base points of the linear system $\mathcal{H}$, and the third one determines the y–coordinate of the free intersection point depending rationally on t.

Similarly, the non–constant factors of $\operatorname{res}_y(f(x,y), h(x,y,t))$ are

$$x^3,$$
$$(256t^4 + 32t^2 + 1)x + 1024t^3.$$

The first factor corresponds to the affine base points of the linear system $\mathcal{H}$, and the second one determines the x–coordinate of the free intersection point depending rationally on t.

So we have found a rational parametrization of $\mathcal{C}$, namely

$$x(t) = \frac{-1024t^3}{256t^4 + 32t^2 + 1}\,, \quad y(t) = \frac{-2048t^4 + 128t^2}{256t^4 + 32t^2 + 1}\,.$$

In the previous example we were lucky enough to find a rational simple point on the curve, allowing us to determine a rational parametrization over the field of definition $\mathbb{Q}$. In fact, there are methods for determining whether a curve of genus 0 has rational simple points, and if so find one. We cannot go into more details here, but we refer the reader to [SeWi97].

From the work of Noether, Hilbert, and Hurwitz we know that it is possible to parametrize any curve $\mathcal{C}$ of genus 0 over the field of definition K, if $\deg(\mathcal{C})$ is odd, and over some quadratic extension of K, if $\deg(\mathcal{C})$ is even. An algorithm which actually achieves this optimal field of parametrization is presented in [SeWi97]. Moreover, if the field of definition is $\mathbb{Q}$, we can also decide if the curve can be parametrized over $\mathbb{R}$, and if so, compute a parametrization over $\mathbb{R}$.

Space curves can be handled by projecting them to a plane along a suitable axis, parametrizing the plane curve, and inverting the projection.