# Algebraic Differential Equations

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# Abstract

Consider an algebraic ordinary differential equation (AODE), i.e. a polynomial relation between the unknown function and its derivatives. This polynomial defines an algebraic hypersurface. By considering rational parametrizations of this hypersurface, we can decide the rational solvability of the given AODE, and in fact compute the general rational solution. This method depends crucially on curve and surface parametrization and the determination of rational invariant algebraic curves.

We also discuss the extension of this method to non-rational solutions, and to partial differential equations.

# Outline

# Solving ADEs — The problem

An algebraic ordinary differential equation (AODE) is given by

$$F(x, y, y', \dots, y^{(n)}) = 0 \ ,$$

where $F$ is a differential polynomial in $K[x]\{y\}$ with $K$ being a differential field and the derivation $'$ being $\frac{d}{dx}$.
Such an AODE is autonomous iff $F \in K\{y\}$.

The radical differential ideal $\{F\}$ can be decomposed

$$\{F\} \ = \ \underbrace{(\{F\} : S)}_{general\ \ component} \ \cap \ \underbrace{\{F, S\}}_{singular\ \ component} \ ,$$

where $S$ is the separant of $F$ (derivative of $F$ w.r.t. $y^{(n)}$).
If $F$ is irreducible, $\{F\} : S$ is a prime differential ideal; its generic zero is called a general solution of the AODE $F(x, y, y', \dots, y^{(n)}) = 0$.

J.F. Ritt, *Differential Algebra* (1950)
E. Hubert, The general solution of an ODE, *Proc. ISSAC 1996*

**Problem: Rational general solution of AODE of order 1**

given: an AODE $F(x, y, y') = 0$, $F$ irreducible in $\overline{\mathbb{Q}}[x, y, y']$

decide: does this AODE have a rational general solution

find: if so, find it

**Example:** $F \equiv y'^2 + 3y' - 2y - 3x = 0$.
general solution: $y = \frac{1}{2}((x + c)^2 + 3c)$, where $c$ is an arbitrary constant.
The separant of $F$ is $S = 2y' + 3$. So the singular solution of $F$ is $y = -\frac{3}{2}x - \frac{9}{8}$.

# Rational parametrizations

An algebraic variety $\mathcal{V}$ is the zero locus of a (finite) set of polynomials $F$, or of the ideal $I = \langle F \rangle$.
A rational parametrization of $\mathcal{V}$ is a rational map $\mathcal{P}$ from a full (affine, projective) space covering $\mathcal{V}$; i.e. $\mathcal{V} = \overline{\mathrm{im}(\mathcal{P})}$ (Zariski closure).
A variety having a rational parametrization is called unirational; and rational if $\mathcal{P}$ has a rational inverse.

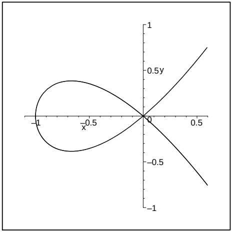
The singular cubic

$$y^2 - x^3 - x^2 = 0$$

has the rational, in fact polynomial, parametrization

$$x(t) = t^2 - 1, \quad y(t) = t^3 - t \ .$$

So this is a unirational curve.



- a parametrization of a variety is a generic point or generic zero of the variety; i.e. a polynomial vanishes on the variety if and only if it vanishes on this generic point
- so only irreducible varieties can be rational
- a rationally invertible parametrization $\mathcal{P}$ is called a proper parametrization;
  every rational curve or surface has a proper parametrization (Lüroth, Castelnuovo); but not so in higher dimensions

For details on parametrizations of algebraic curves we refer to
J.R. Sendra, F. Winkler, S. Pérez-Díaz,
*Rational Algebraic Curves – A Computer Algebra Approach*,
Springer-Verlag Heidelberg (2008)

# Rational solutions of AODEs of order 1

The autonomous case $F(y, y') = 0$

First we concentrate on algebraic and geometric questions:

- A rational solution of $F(y, y') = 0$ corresponds to a proper (because of the degree bounds) rational parametrization of the algebraic curve $F(y, z) = 0$.
- Conversely, from a proper rational parametrization $(f(x), g(x))$ of the curve $F(y, z) = 0$ we get a rational solution of $F(y, y') = 0$ if and only if there is a linear rational function $T(x)$ such that $f(T(x))' = g(T(x))$.

References:

- R. Feng, X-S. Gao, Proc. ISSAC 2004
- R. Feng, X-S. Gao, JSC 41 (2006)

based on degree bounds derived in

- J.R. Sendra, F. Winkler, Comp.Aided Geom.Design 18 (2001)

## The general (non-autonomous) case $F(x, y, y') = 0$

It is now natural to assume that the solution surface $F(x, y, z) = 0$ is a rational algebraic surface, i.e. rationally parametrized by

$$\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t)).$$

Then $\mathcal{P}(s, t)$ creates a rational solution of $F(x, y, y') = 0$ if and only if we can find two rational functions $s(x)$ and $t(x)$ which solve the following associated system:

$$s' = \frac{f_1(s, t)}{g(s, t)}, \quad t' = \frac{f_2(s, t)}{g(s, t)}, \tag{1}$$

where $f_1(s, t), f_2(s, t), g(s, t)$ are rational functions in $s, t$.

The construction of the associated system and the following theorem can be found in
L.X.C. Ngô, F. Winkler, JSC, 45/12 (2010)

## Theorem

*There is a one-to-one correspondence between rational general solutions of the AODE $F(x, y, y') = 0$ and rational general solutions of its associated system.*

The associated system is

- autonomous
- of order 1
- of degree 1 in the derivatives of the parameters

## Lemma

*Every non-trivial rational solution of the associated system corresponds to a rational invariant algebraic curve, i.e. a curve $G(s, t) = 0$ satisfying $G_s \cdot N_1 M_2 + G_t \cdot N_2 M_1 \in \langle G \rangle$.*

In the generic case (assoc. system has no dicritical points) there is an upper bound for irreducible invariant algebraic curves.

Rational invariant algebraic curves create candidates for rational solutions of the associated system, and therefore of the original AODE;
if we can find a linear transformation s.t. the derivation of the first component is equal to the second.

L.X.C. Ngô, F. Winkler, JSC 46/10, (2011)

**Example:** Consider the differential equation

$$F(x, y, y') \equiv y'^2 + 3y' - 2y - 3x = 0 \ .$$

The solution surface $z^2 + 3z - 2y - 3x = 0$ has the parametrization

$$\mathcal{P}(s,t) = \left( \frac{t}{s} + \frac{2s + t^2}{s^2}, -\frac{1}{s} - \frac{2s + t^2}{s^2}, \frac{t}{s} \right).$$

This is a proper parametrization and its associated system is

$$s' = st, \quad t' = s + t^2 \ .$$

Irreducible invariant algebraic curves of the system are:

$$G(s,t) = s, \quad G(s,t) = t^2 + 2s, \quad G(s,t) = s^2 + ct^2 + 2cs$$

The third algebraic curve $s^2 + ct^2 + 2cs = 0$ depends on a transcendental parameter $c$. It can be parametrized by

$$\mathcal{Q}(x) = \left( -\frac{2c}{1+cx^2}, -\frac{2cx}{1+cx^2} \right).$$

Running Step 5 in RATSOLVE, the differential equation defining the reparametrization is $T' = 1$. Hence $T(x) = x$. So the rational solution in this case is

$$s(x) = -\frac{2c}{1+cx^2}, \quad t(x) = -\frac{2cx}{1+cx^2}.$$

Since $G(s,t)$ contains a transcendental constant, the above solution is a rational general solution of the associated system. Therefore, the rational general solution of $F(x,y,y') = 0$ is

$$y = \frac{1}{2}x^2 + \frac{1}{c}x + \frac{1}{2c^2} + \frac{3}{2c},$$

which, after a change of parameter, can be written as

$$y = \frac{1}{2}(x^2 + 2cx + c^2 + 3c).$$

Recently we have been able to derive an algorithm for deciding the existence of a strong rational general solution, i.e. a general solution in $K(c, x)$ (also rational in the constant $c$):
the curve

$$\mathcal{C} = \{(a_1, a_2) \in \mathbb{A}^2(\overline{K(x)}) \mid F(x, a_1, a_2) = 0 \}$$

must have genus 0.

If $F = 0$ has a strong rational general solution, then it has an associated Riccati equation

$$\omega' = a_0(x) + a_1(x)\omega + a_2(x)\omega^2 \ .$$

Existence of rational general solution can be decided.

So we have a full decision algorithm for finding rational general solutions of strongly parametrizable AODEs.
Almost all first-order AODEs in the collection of Kamke are in this class. These cover 64 percent of all the first-order ODEs in Kamke.

**Example** 1.537 in Kamke: Consider the differential equation

$$\begin{aligned} F(x, y, y') &= x^3 y'^3 - 3x^2 y y'^2 + (x^6 + 3xy^2) y' - y^3 - 2x^5 y \\ &= (xy' - y)^3 + x^6 y' - 2x^5 y = 0 \ . \end{aligned}$$

The associated curve defined by $F(x, y, z) = 0$ can be parametrized by

$$\mathcal{P}(t) = \left( -\frac{t^3 x^5 - t^2 x^6 + (t - x)^3}{t^3 x^5}, -\frac{2t^3 x^5 - 2t^2 x^6 + (t - x)^3}{t^3 x^6} \right) \ .$$

Therefore, the associated differential (Riccati) equation with respect to $\mathcal{P}$ is

$$\omega' = \frac{1}{x^2} \omega (2\omega - x) \ .$$

This Riccati equation has the rational general solution

$$\omega(x) = \frac{x}{1 + cx^2} \ .$$

Hence, the original AODE $F(x, y, y') = 0$ has the rational general solution

$$y(x) = cx(x + c^2) \ .$$

# Classification of AODEs / differential orbits

- consider a group of transformations leaving the associated system of an AODE invariant; orbits w.r.t. such a transformation group contain AODEs of equal complexity in terms of determining rational solutions
- we have studied affine and birational transformations
- it turns out that being autonomous is not an intrinsic property of an AODE; certain classes contain both autonomous and non-autonomous AODEs

# Extension to non-rational solutions

(G.Grasegger, PhD Thesis)

(a) $y^8y' - y^5 - y' = 0$:
parametrization: $(\frac{1}{t}, \frac{t^3}{1-t^8})$,
radical solution: $y(x) = -\left(2(x+c) - \sqrt{-1+4(x+c)^2}\right)^{-1/4}$

(b) $4y^7 - 4y^5 - y^3 - 2y' - 8y^2y' + 8y^4y' + 8yy'^2 = 0$: (genus 1)
parametrization: $\left(\frac{1}{t}, \frac{-4+4t^2+t^4}{t(4t^2-4t^4-t^6-\sqrt{t^{12}+8t^{10}+16t^8-16t^4})}\right)$
radical solution: $y(x) = -\frac{\sqrt{1+c+x}}{\sqrt{1+(c+x)^2}}$

(c) $y^3 + y^2 + y'^2 = 0$:
parametrization: $(-1-t^2, t(-1-t^2))$,
trigonometric solution: $y(x) = -1 - \tan\left(\frac{x+c}{2}\right)^2$

(d) $y^2 + y'^2 + 2yy' + y = 0$:
parametrization: $\left(-\frac{1}{(1+t)^2}, -\frac{t}{(1+t)^2}\right)$
exponential solution: $y(x) = -e^{-x}(-1+e^{x/2})^2$

# Publications

- L.X.C. Ngô, F. Winkler, *Rational general solutions of first order non-autonomous parametrizable ODEs*, *J. Symbolic Computation*, **45/12**, 1426–1441, 2010.
- L.X.C. Ngô, F. Winkler, *Rational general solutions of planar rational systems of autonomous ODEs*, *J. Symbolic Computation*, **46/10**, 1173–1186, 2011.
- L.X.C. Ngô, F. Winkler, *Rational general solutions of parametrizable AODEs*, *Publ. Math. Debrecen*, **79/3-4**, 573–587, 2011.
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- T.N. Vo, F. Winkler, *Algebraic general solutions of first-order algebraic ODEs*, in *Proc. CASC-2015*, 479–492, V.P.Gerdt et al. (eds.) (2015).
- G. Grasegger, A. Lastra, J.R. Sendra, F. Winkler, *A solution method for autonomous first-order algebraic partial differential equations*, *J. Computational and Applied Mathematics*, **300**, 119-133 (2016).
- N.T. Vo, G. Grasegger, F. Winkler, *Deciding the existence of rational general solutions for first-order algebraic ODEs*, submitted

# Conclusion

- we can decide whether an AODE has a strong general rational solution,
  and if it has, we can determine such a general rational solution
- we have a generalization of the solution method for rational solutions to other types of solutions; but not yet a decision procedure
- currently we are extending the method to algebraic partial differential equations
- we have ideas for finding ALL rational solutions of an AODE

Thank you for your attention!