# Chapter 1

# Introduction and overview

Consider an algebraic ordinary differential equation (AODE), i.e. a polynomial relation between the unknown function and its derivatives. This polynomial defines an algebraic hypersurface. By considering rational parametrizations of this hypersurface, we can decide the rational solvability of the given AODE, and in fact compute the general rational solution. This method depends crucially on curve and surface parametrization and the determination of rational invariant algebraic curves.

We also discuss the extension of this method to non-rational solutions, and to partial differential equations.

## 1.1 Symbolic solution of ADEs — The problem

An **algebraic ordinary differential equation (AODE)** is given by

$$F(x, y, y', \ldots, y^{(n)}) = 0\ ,$$

where $F$ is a differential polynomial in $K[x]\{y\}$ with $K$ being a differential field and the derivation $'$ being $\frac{d}{dx}$. Such an AODE is **autonomous** iff $F \in K\{y\}$.

The radical differential ideal $\{F\}$ can be decomposed

$$\{F\} \;=\; \underbrace{(\{F\} : S)}_{general\ \ component} \;\cap\; \underbrace{\{F, S\}}_{singular\ \ component}\ ,$$

where $S$ is the separant of $F$ (derivative of $F$ w.r.t. $y^{(n)}$) (compare [Ritt50]).
If $F$ is irreducible, $\{F\} : S$ is a prime differential ideal; its generic zero is called a **general solution** of the AODE $F(x, y, y', \ldots, y^{(n)}) = 0$.

**Problem: general solution of AODE of order 1**

given: an AODE $F(x, y, \ldots, y^{(n)}) = 0$, $F$ irreducible in $\overline{\mathbb{Q}}[x, y, \ldots, y^n]$

decide: does this AODE have a general solution in a predefined class of functions?

find: if so, find it!

**Example:** Consider the AODE $F(x, y, y') = y'^2 + 3y' - 2y - 3x = 0$. A general solution for this AODE is $y = \frac{1}{2}((x+c)^2 + 3c)$, where $c$ is an arbitrary constant.
The separant of $F$ is $\frac{dF}{dy'} = 2y' + 3$. So the singular solution of $F$ is $y = -\frac{3}{2}x - \frac{9}{8}$. □

## 1.2 Rational parametrizations

An **algebraic variety** $\mathcal{V}$ is the zero locus of a (finite) set of polynomials $F$, or of the ideal $I = \langle F \rangle$.
A **rational parametrization** of $\mathcal{V}$ is a rational map $\mathcal{P}$ from a full (affine, projective) space covering $\mathcal{V}$; i.e. $\mathcal{V} = \overline{\mathrm{im}(\mathcal{P})}$ (Zariski closure).
A variety having a rational parametrization is called **unirational**, and **rational** if $\mathcal{P}$ has a rational inverse.

**Example:** The singular cubic (node)

$$y^2 - x^3 - x^2 = 0$$

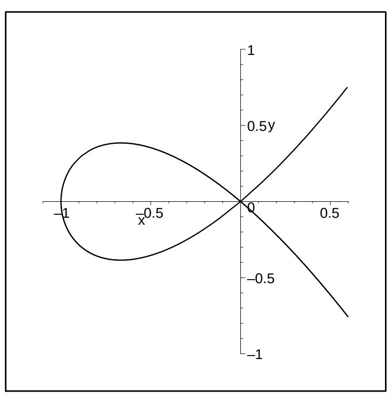


Figure 1.1: node

has the rational, in fact polynomial, parametrization

$$x(t) = t^2 - 1, \quad y(t) = t^3 - t \ .$$

So this is a unirational curve. □

**Remark:**

- A parametrization of a variety is a **generic point** or **generic zero** of the variety; i.e. a polynomial vanishes on the variety if and only if it vanishes on this generic point.
- so only irreducible varieties can be rational
- a rationally invertible parametrization $\mathcal{P}$ is called a **proper** parametrization; every rational curve or surface has a proper parametrization (Lüroth, Castelnuovo); but not so in higher dimensions

For details on parametrizations of algebraic curves we refer to [SWP08].

## 1.3 Rational solutions of AODEs of order 1

The autonomous case $F(y, y') = 0$

First we concentrate on algebraic and geometric questions:

- A rational solution of $F(y, y') = 0$ corresponds to a proper (because of the degree bounds) rational parametrization of the algebraic curve $F(y, z) = 0$.
- Conversely, from a proper rational parametrization $(f(x), g(x))$ of the curve $F(y, z) = 0$ we get a rational solution of $F(y, y') = 0$ if and only if there is a linear rational function $T(x)$ such that $f(T(x))' = g(T(x))$.

So we first check whether the corresponding algebraic curve has a rational parametrization. If so, we try to transform this parametrization into another one, such that the second component is the derivative of the first.
This idea first appeared in [FeG04], [FeG06], based on degree bounds derived in [SeW01].

The general (non-autonomous) case $F(x, y, y') = 0$

Consider the AODE

$$F(x, y, y') = 0 \ . \tag{1.1}$$

It is now natural to assume that the **solution surface** $F(x, y, z) = 0$ is a rational algebraic surface, i.e. rationally parametrized by

$$\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t)).$$

Then $\mathcal{P}(s,t)$ creates a rational solution of $F(x,y,y')=0$ if and only if we can find two rational functions $s(x)$ and $t(x)$ which solve the following **associated system**:

$$s' = \frac{M_1(s,t)}{N_1(s,t)}, \quad t' = \frac{M_2(s,t)}{N_2(s,t)}, \tag{1.2}$$

where $M_1(s,t), M_2(s,t), N_1(s,t), N_2(s,t)$ are certain rational functions in $s,t$. The construction of the associated system and the following theorem can be found in [NgW10].

**Theorem:** *There is a one-to-one correspondence between rational general solutions of the AODE $F(x,y,y')=0$ and rational general solutions of its associated system.*
*The associated system is*

- *autonomous*
- *of order 1*
- *of degree 1 in the derivatives of the parameters*

**Lemma:** *Every non-trivial rational solution of the associated system corresponds to a rational invariant algebraic curve, i.e. a curve $G(s,t)=0$ satisfying*

$$G_s \cdot N_1 M_2 + G_t \cdot N_2 M_1 \in \langle G \rangle \ .$$

In the generic case (assoc. system has no dicritical points) there is an upper bound for irreducible invariant algebraic curves. Rational invariant algebraic curves create candidates for rational solutions of the associated system, and therefore of the original AODE. This is described in [NgW11].

**Theorem:** *Solutions of the original DE (1.1) and the system (1.2) correspond to each other; in particular, if $(s(x),t(x))$ is a general rational solution of (1.2) and $c = \chi_1(s(x),t(x)) - x$, then $y = \chi_2(s(x-c),t(x-c))$ is a rational general solution of (1.1).*

**Example:** Consider the differential equation

$$F(x,y,y') \equiv y'^2 + 3y' - 2y - 3x = 0 \ .$$

The solution surface $z^2 + 3z - 2y - 3x = 0$ has the parametrization

$$\mathcal{P}(s,t) = \left( \frac{t}{s} + \frac{2s+t^2}{s^2}, -\frac{1}{s} - \frac{2s+t^2}{s^2}, \frac{t}{s} \right).$$

This is a proper parametrization and its associated system is

$$s' = st, \quad t' = s + t^2 \ .$$

Irreducible invariant algebraic curves of the system are:

$$G(s,t) = s, \quad G(s,t) = t^2 + 2s, \quad G(s,t) = s^2 + ct^2 + 2cs \ .$$

The third algebraic curve $s^2 + ct^2 + 2cs = 0$ depends on a transcendental parameter $c$. It can be parametrized by

$$\mathcal{Q}(x) = \left( -\frac{2c}{1+cx^2}, -\frac{2cx}{1+cx^2} \right) \ ,$$

and a rational solution in this case is

$$s(x) = -\frac{2c}{1+cx^2}, \ \ t(x) = -\frac{2cx}{1+cx^2}.$$

Since $G(s,t)$ contains a transcendental constant, the above solution is a rational general solution of the associated system. Therefore, a rational general solution of $F(x,y,y') = 0$ is

$$y = \frac{1}{2}x^2 + \frac{1}{c}x + \frac{1}{2c^2} + \frac{3}{2c},$$

which, after a change of parameter, can be written as

$$y = \frac{1}{2}((x+c)^2 + 3c) \ . \qquad \square$$

Recently in [VGW18] we have been able to derive an algorithm for deciding the existence of a **strong rational general solution**, i.e. a general solution in $K(c,x)$ (also rational in the constant $c$):
the curve

$$\mathcal{C} = \{(a_1, a_2) \in \mathbb{A}^2(\overline{K(x)}) \mid F(x, a_1, a_2) = 0 \ \}$$

must have genus 0.

If $F = 0$ has a strong rational general solution, then it has an associated Riccati equation

$$\omega' = a_0(x) + a_1(x)\omega + a_2(x)\omega^2 \ .$$

Existence of rational general solution can be decided. So we have a full decision algorithm for finding rational general solutions of strongly parametrizable AODEs. Almost all first-order AODEs in the collection of Kamke are in this class. These cover 64 percent of all the first-order ODEs in Kamke.

**Example:** (1.537 in Kamke) *Consider the differential equation*

$$\begin{aligned} F(x,y,y') &= x^3y'^3 - 3x^2yy'^2 + (x^6 + 3xy^2)y' - y^3 - 2x^5y \\ &= (xy' - y)^3 + x^6y' - 2x^5y = 0 \ . \end{aligned}$$

*The associated curve defined by $F(x,y,z)=0$ can be parametrized by*

$$\mathcal{P}(t)=\left(-\frac{t^3x^5-t^2x^6+(t-x)^3}{t^3x^5},-\frac{2t^3x^5-2t^2x^6+(t-x)^3}{t^3x^6}\right)\ .$$

*Therefore, the associated differential (Riccati) equation with respect to $\mathcal{P}$ is*

$$\omega'=\frac{1}{x^2}\omega(2\omega-x)\ .$$

*This Riccati equation has the rational general solution*

$$\omega(x)=\frac{x}{1+cx^2}\ .$$

*Hence, the original AODE $F(x,y,y')=0$ has the rational general solution*

$$y(x)=cx(x+c^2)\ . \qquad \square$$

## 1.4 Extension to non-rational solutions

Not only rational solutions, but also other types of solutions can be computed with a similar approach. But for these other types we have no decision algorithm; see [G.Grasegger, PhD Thesis, 2015].

(a) $y^8y'-y^5-y'=0$ :
parametrization: $(\frac{1}{t},\frac{t^3}{1-t^8})$,
radical solution: $y(x)=-\Big(2(x+c)-\sqrt{-1+4(x+c)^2}\Big)^{-1/4}$

(b) $4y^7-4y^5-y^3-2y'-8y^2y'+8y^4y'+8yy'^2=0$ : (genus 1)
parametrization: $\left(\frac{1}{t},\frac{-4+4t^2+t^4}{t(4t^2-4t^4-t^6-\sqrt{t^{12}+8t^{10}+16t^8-16t^4})}\right)$
radical solution: $y(x)=-\frac{\sqrt{1+c+x}}{\sqrt{1+(c+x)^2}}$

(c) $y^3+y^2+y'^2=0$ :
parametrization: $(-1-t^2,t(-1-t^2))$,
trigonometric solution: $y(x)=-1-\tan\left(\frac{x+c}{2}\right)^2$

(d) $y^2+y'^2+2yy'+y=0$ :
parametrization: $\left(-\frac{1}{(1+t)^2},-\frac{t}{(1+t)^2}\right)$
exponential solution: $y(x)=-e^{-x}(-1+e^{x/2})^2$