ROBOT KINEMATICS

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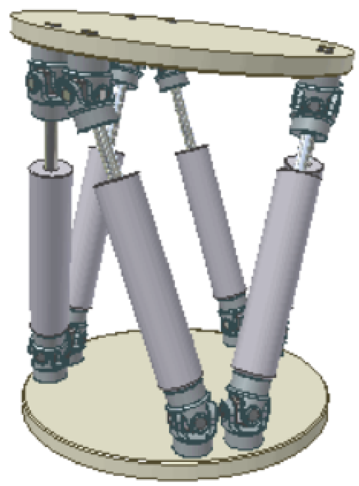
Abstract. One of the first applications of Groebner bases was in kinematics. To be more specific, the question was how many configurations of the actuators of a Stewart platform can be found such that the platform is in a particular position.



In this small excerpt we want to raise natural questions from robot kinematics which can be partly answered by using Groebner bases. We will distinguish between the so called forward and inverse kinematics.

1. Introduction

To treat the space of configurations of a robot geometrically , we make some assumptions for simplification. We consider robots constructed from rigid links connected by joints. The links or segments are connected in series, where one end is in fixed position and the other end is the hand of the robot, and all segments lie on a plane. Since the

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segments are rigid, motions can only be performed by the joints. Here we distinguish between revolute joints, performing rotation around the joint, and prismatic joints, performing translation along an axis.

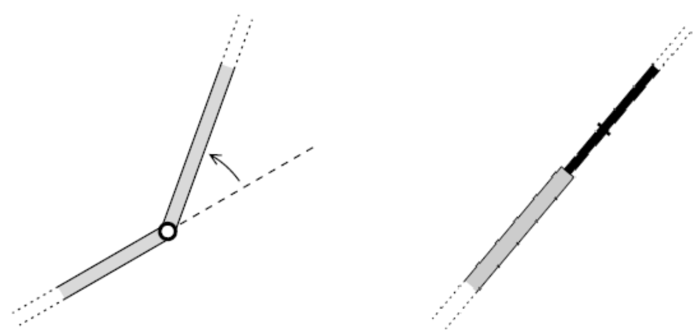
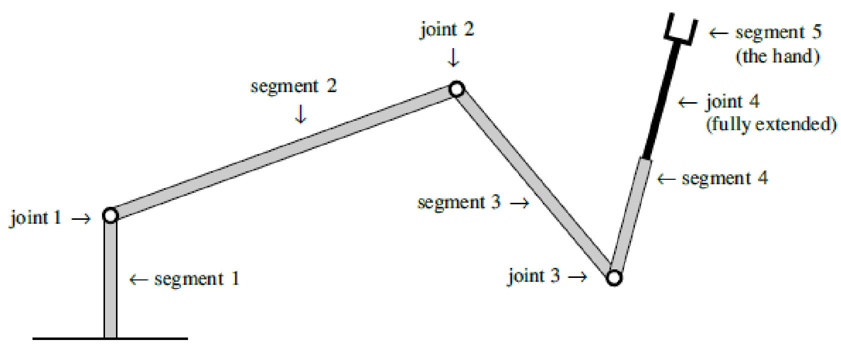


FIGURE 1. revolute joint (left) and prismatic joint (right).

For simplicity, we number segments and joints of a robot in increasing order out from the fixed end to the hand.



The position of a revolute joint between segments i and $i+1$ can be described by the angle (we do that counterclockwise). Similarly, the position of a prismatic joint can be specified by the extension. If the joint positions can be set independent, then the possible settings of the robot can be described by

$$\mathcal{J} = S_1 \times \cdots \times S_n \times I_1 \times \cdots \times I_p,$$

where the S_j are subsets of $[0, 2\pi]$ describing the revolute joints and I_j are some real intervals describing the prismatic joints. We call $\mathcal{J}$ the *joint space* of the robot. Moreover, the hand is described by the position and an angle specifying the orientation, so the *configuration space* of the robot's hand is

$$\mathcal{C} = U \times V \subset \mathbb{R}^2 \times [0, 2\pi].$$

Each element in the joint space uniquely determines the hand of the robot and we define this mapping, also *movement function* called, as

$$f \ : \ \mathcal{J} \rightarrow \mathcal{C}.$$

The two basic problems we are dealing with are the following:

- *Forward Kinematic Problem:* Can we explicitly give the movement function f and find all possible hand configurations, namely $f(\mathcal{J})$?
- *Inverse Kinematic Problem:* Given a hand configuration $c \in \mathcal{C}$, can we determine one or all joint settings to realize c? In other words, can we find $f^{-1}(\{c\})$?

In order to handle the inverse kinematic problem we try to solve the forward kinematic problem first. We also note that in the inverse kinematic problem we usually want to compute all possible solutions, because in real world some joint settings might not be possible or less practical.

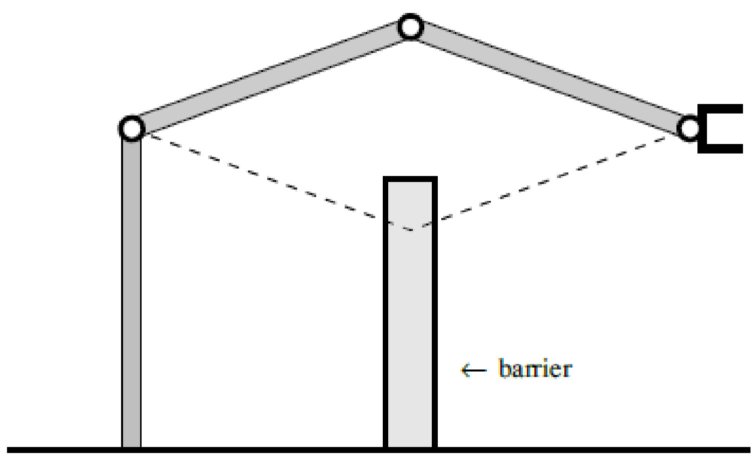
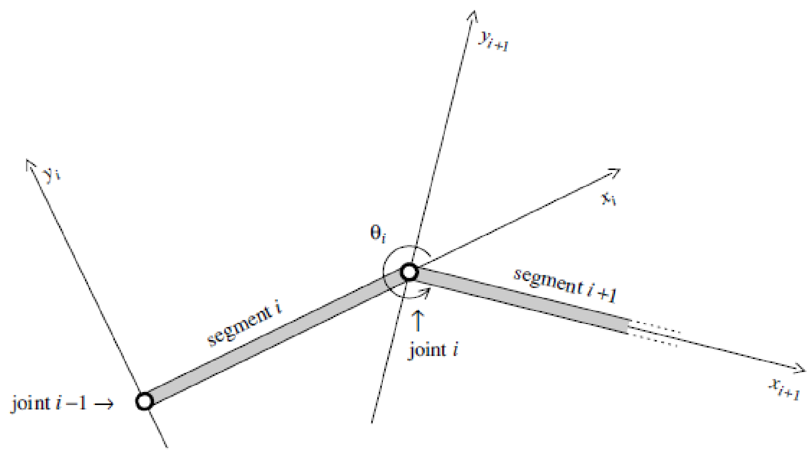


FIGURE 2. Two possible joint settings where one is not possible due to an obstacle.

2. Forward Kinematic Problem

In this section we present methods to explicitly write down the movement function. For our robot we introduce a global coordinate system (x_1, y_1) placed at joint 1. Additionally we define for every revolute joint i a local coordinate system (x_{i+1}, y_{i+1}) with origin placed at joint i and the positive x_{i+1}-axis along the direction of segment $i+1$. We note that for every $i \geq 2$ the (x_i, y_i) coordinates of joint i are $(l_i, 0)$, where l_i is the length of segment i (possibly plus the extension of prismatic joints in between).



From Linear Algebra we know how to express the coordinates of a point (a_{i+1}, b_{i+1}), given in the coordinate system (x_{i+1}, y_{i+1}), in the coordinate system (x_i, y_i), namely by multiplying with a rotation matrix and adding a translation vector:

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) \end{pmatrix} \cdot \begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix} + \begin{pmatrix} l_i \\ 0 \end{pmatrix},$$

or written equivalently as

$$\begin{pmatrix} a_i \\ b_i \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i) & l_i \\ \sin(\theta_i) & \cos(\theta_i) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix} =: A_i \cdot \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix}.$$

Example 1: We now want to particularly consider the case where n revolute and no prismatic joints occur. For simplicity we may do some calculations with $n = 3$. Doing this matrix-multiplication above several times, we can express the hand in the global coordinate system:

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = A_1 \cdots A_n \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

By expanding this equation we obtain

$$f(\theta_1, \ldots, \theta_n) = \begin{pmatrix} \sum_{i=1}^{n-1} l_{i+1} \cos(\sum_{j=1}^{i} \theta_j) \\ \sum_{i=1}^{n-1} l_{i+1} \sin(\sum_{j=1}^{i} \theta_j) \\ \sum_{i=1}^{n} \theta_i \end{pmatrix}. \tag{2.1}$$

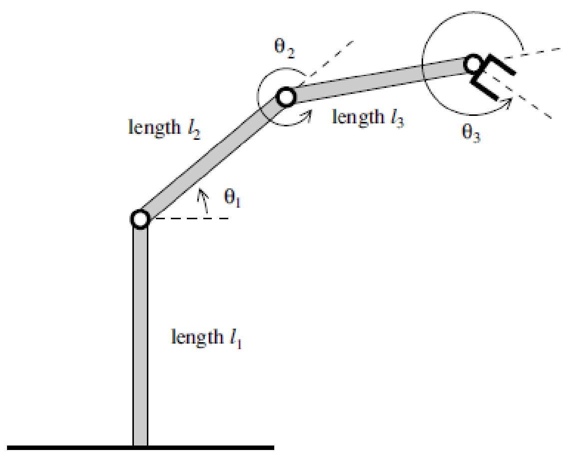


FIGURE 3. 3 revolute and no prismatic joints.

We now want to find other ways of specifying f. By using the implicit description of the circle, the domain $\mathcal{J}$ can also be written as the product of a variety with real intervals

$$\mathcal{J} = V(\{x_i^2 + y_i^2 - 1 \mid 1 \leq i \leq n\}) \times I_1 \times \cdots \times I_p,$$

and changing the movement function by the substitutions

$$c_i = \cos(\theta_i), \;\; s_i = \sin(\theta_i)$$

and using trigonometric formulas we can write the hand position as a polynomial mapping on $\mathcal{J}$.

Example 1 (continued): The first two components of the movement function can be written as

$$\begin{pmatrix} l_3(c_1c_2 - s_1s_2) + l_2c_1 \\ l_3(s_1c_2 + s_2c_1) + l_2s_1 \end{pmatrix}. \tag{2.2}$$

The third component of the movement function, namely the orientation of the hand, cannot be written as a polynomial in this way. Since $\mathcal{J}$ is now given as a variety, another possibility to represent the movement function is by finding a rational parametrization. For the circle we know that

$$c_i = \frac{1-t_i^2}{1+t_i^2}, \; s_i = \frac{2t_i}{1+t_i^2} \tag{2.3}$$

is such a rational parametrization. Now we can compose the polynomial representation of f with these parametrizations and the hand position is given rationally. This method decreases the number of variables again (Note that we had a dependency on 2 variables in the first two components of f in (2.1) and on 4 variables in (2.2)) and no constraints in $\mathcal{J}$ are needed. We note that there is no possibility to give a surjective rational parametrization of the circle. For our choice (2.3), $(-1, 0)$ gets only reached for t_i goes to infinity. This implies that for certain hand configurations we have to use very high values for the t_i's or even do not reach it.

Example 1 (continued): By using (2.3), the hand position can be described as a rational function on $\mathbb{R}^2$, namely by (t_1, t_2) maps to

$$\begin{pmatrix} \frac{l_3(1-t_1^2)(1+t_2^2)+l_2(1-4t_1t_2-t_2^2+t_1^2(-1+t_2^2))}{(1+t_1^2)(1+t_2^2)} \\ \frac{2(l_3t_1(1+t_2^2)+l_2(t_1+t_2-t_1^2t_2-t_1t_2^2))}{(1+t_1^2)(1+t_2^2)} \end{pmatrix}. \tag{2.4}$$

Example 2: We shortly want to discuss the case where one prismatic joint is right before the hand.

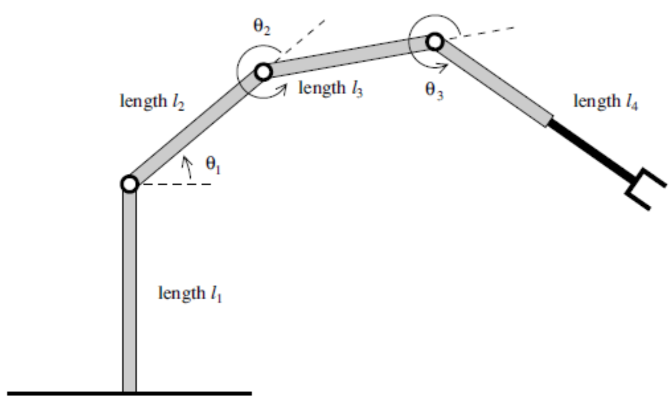


FIGURE 4. 3 revolute and 1 prismatic joints.

Similarly as in (2.1) we have

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = A_1 \cdots A_n \begin{pmatrix} l_{n+1} \\ 0 \\ 1 \end{pmatrix}$$

and therefore,

$$f(\theta_1, \dots, \theta_n, l_{n+1}) = \begin{pmatrix} \sum_{i=1}^n l_{i+1} \cos(\sum_{j=1}^i \theta_j) \\ \sum_{i=1}^n l_{i+1} \sin(\sum_{j=1}^i \theta_j) \\ \sum_{i=1}^n \theta_i \end{pmatrix}.$$

In the case where $n = 3$, the first two components of the movement function can be written as

$$\begin{pmatrix} l_4(c_1(c_2c_3 - s_2s_3) - s_1(c_2s_3 + c_3s_2)) + l_3(c_1c_2 - s_1s_2) + l_2c_1 \\ l_4(s_1(c_2c_3 - s_2s_3) + c_1(c_2s_3 + c_3s_2)) + l_3(s_1c_2 + s_2c_1) + l_2s_1 \end{pmatrix}$$

and we choose $\mathcal{J} = V(x_1^2 + y_1^2 - 1, x_2^2 + y_2^2 - 1, x_3^2 + y_3^2 - 1) \times [m_1, m_2]$ for some $m_1 < m_2 \in \mathbb{R}$. Further reasonings can be made similarly to Example 1.

3. Inverse Kinematic Problem

In this section we continue Example 1 from before and discuss whether it is possible to find a joint setting in order to realize a given point $(a, b) = (x_1, y_1)$ and orientation α of the hand. In the affirmative case we want to find all such configurations. In other words, we want to compute $f^{-1}(\{(a, b, \alpha)\})$.

Example 1 (continued): Every hand orientation can be realized, since θ_3 is independent of the other variables and we can set it to $\theta_3 = \alpha - \theta_1 - \theta_2$. Hence, we focus on placing the hand at (a, b). The polynomial system

$$\begin{aligned}
a &= l_3(c_1c_2 - s_1s_2) + l_2c_1,\\
b &= l_3(c_1s_2 + c_2s_1) + l_2s_1,\\
0 &= c_1^2 + s_1^2 - 1,\\
0 &= c_2^2 + s_2^2 - 1
\end{aligned}$$

can be derived from (2.2). We compute a Groebner basis with the lex order $c_2 > s_2 > c_1 > s_1$ in $\mathbb{R}(a, b, l_2, l_3)[s_1, c_1, s_2, c_2]$ and keep a, b, l_2, l_3 as parameters to obtain:

$$G = \left\{ \begin{array}{l}
c_2 - \dfrac{a^2+b^2-l_2^2-l_3^2}{2l_2l_3},\ s_2 + \dfrac{a^2+b^2}{al_3}s_1 - \dfrac{a^2b+b^3+b(l_2^2-l_3^2)}{2al_2l_3},\\
c_1 + \dfrac{b}{a}s_1 - \dfrac{a^2+b^2+l_2^2-l_3^2}{2al_2}, s_1^2 - \dfrac{a^2b+b^3+b(l_2^2-l_3^2)}{l_2(a^2+b^2)}s_1 + \\
\quad \dfrac{(a^2+b^2)^2+(l_2^2-l_3^2)^2-2a^2(l_2^2+l_3^2)+2b^2(l_2^2-l_3^2)}{4l_2^2(a^2+b^2)}
\end{array} \right\}.$$

As long as we keep the parameters as abstract variables, G remains a Groebner bases. However, if we substitute some real numbers to the parameters- this is called *specialization*- we first have to be careful with denominators and second, the new polynomial system $G|_{\{a,b,l_2,l_3\}} \in \mathbb{R}[s_1, c_1, s_2, c_2]$ does not have to be a Groebner basis anymore. In general, there will be a lower dimensional subspace $W \subset \mathbb{R}^4$ where $G|_{\{a,b,l_2,l_3\}}$ is not a Groebner basis anymore. When W is the empty set, the initial Groebner basis G is called a *comprehensive Grobner bases*, see for example in the appendix of [2].

Example 1 (continued): Let us choose $l_2 = l_3 = 1$. We can either use this specialization in G or recompute the Groebner basis with these values to obtain

$$G|_{\{l_2=1,l_3=1\}} = \left\{ \begin{array}{l} c_2 - \dfrac{a^2+b^2-2}{2}, \\ s_2 + \dfrac{a^2+b^2}{a} s_1 - \dfrac{a^2 b + b^3}{2a}, \\ c_1 + \dfrac{b}{a} s_1 - \dfrac{a^2+b^2}{2a}, \\ s_1^2 - b s_1 + \dfrac{(a^2+b^2)^2 - 4a^2}{4(a^2+b^2)}, \end{array} \right\},$$

which will remain a Groebner basis also for further specialization where $a \neq 0$. Under this assumption we can compute

$$s_1 = \frac{b}{2} \pm \frac{|a|\sqrt{4-(a^2+b^2)}}{2\sqrt{a^2+b^2}},$$

which are two distinct real values for $0 < a^2+b^2 < 4$ and a double root for $a^2+b^2 = 4$. The values for c_1, c_2, s_2 are uniquely determined then. This is exactly what we would have expected from geometrical point of view: The maximum distance between joint 1 and 3 is $l_2 + l_3 = 2$, where this extremal position can be reached only in one way, whereas there are two possibilities to set joint 2 for closer positions of the hand. Now let us consider $a = 0$ (and $b \neq 0$). We first do the specializations and then compute the Groebner basis

$$G|_{\{a=0,l_2=1,l_3=1\}} = \left\{ c_2 - \frac{b^2-2}{2},\ s_2 - bc_1,\ c_1^2 + \frac{b^2-4}{4},\ s_1 - \frac{b}{2} \right\}.$$

Note that the form of this Groebner basis differs a lot from the previous ones. We obtain one solution for s_1 and two distinct real values for c_1 if $|b| < 2$ and a double root if $|b| = 2$.
To summarize, given any point (a, b) for the place of the hand, there are

- infinitely many distinct settings of joint 1 when $a^2 + b^2 = 0$,
- two distinct settings of joint 1 when $a^2 + b^2 < 4$,
- one setting of joint 1 when $a^2 + b^2 = 4$,
- no possible setting of joint 1 when $a^2 + b^2 > 4$.

Let J_f denote the Jacobian matrix of the configuration mapping f. The Jacobian matrix is an $m \times n$ matrix, where $n = \dim(\mathcal{J}), m = \dim(\mathcal{C})$, with rank at most $\min(m, n)$. We call a $c \in \mathcal{J}$ a *kinematic singularity*, if the rank of $J_f(c)$ is strictly less than $\min(m, n)$.

Example 1 (continued): We compute the Jacobian matrix

$$\begin{pmatrix} -\sum_{i=1}^{n-1} l_{i+1}\sin(\sum_{j=1}^{i}\theta_j) & -\sum_{i=2}^{n-1} l_{i+1}\sin(\sum_{j=1}^{i}\theta_j) & \dots & -l_n\sin(\theta_{n-1}) & 0 \\ \sum_{i=1}^{n-1} l_{i+1}\cos(\sum_{j=1}^{i}\theta_j) & \sum_{i=2}^{n-1} l_{i+1}\cos(\sum_{j=1}^{i}\theta_j) & \dots & l_n\cos(\theta_{n-1}) & 0 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}.$$

When all $\theta_i \in \{0, \pi\}$, then the first row is equal to zero and the rank is smaller or equal to two for those c.
When $n = 3$, we are also able to use the determinant of J_f to check their rank:

$$\det(J_f) = \sin(\theta_1 + \theta_2)\cos(\theta_1) - \cos(\theta_1 + \theta_2)\sin(\theta_1) = \sin(\theta_2),$$

which is equal to zero if and only if $\theta_2 \in \{0, \pi\}$. This are exactly the cases where $a^2 + b^2 \in \{0, 4\}$ in the notation of polynomials.

So far we have taken a look which configurations of the hand can be realized. The motion, the process for reaching this configuration, has not been of interest. An important question, however, is to find a path from an initial hand configuration to a new desired configuration and try to minimize the total joint movement. Kinematic singularities play in important role in motion planning. Let us assume that $j(t)$ is a joint space path with image $c(t) = f(j(t))$. Then we obtain by differentiation with respect to t

$$c'(t) = J_f(j(t)) \cdot j'(t) \tag{3.1}$$

We can interpret $j'(t)$ as the *joint space velocity* and $c'(t)$ as the velocity of our configuration space path. If at time t_0 the joint space path $j(t_0)$ passes through a kinematic singularity, (3.1) may not have a solution for $j'(t_0)$. This means that there is no smooth joint path $j(t)$ corresponding to a $c(t)$ moving in certain directions.

Example 1 (continued): Let $\theta_1 = 0$ and $\theta_2 = \pi$, then

$$c'(t_0) = J_f(t_0) \cdot j'(t_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \cdot j'(t_0).$$

If we want to move the hand in the x_1-direction, the first component of $c'(t_0)$ should be non-zero which is not possible for any $j'(t_0)$!

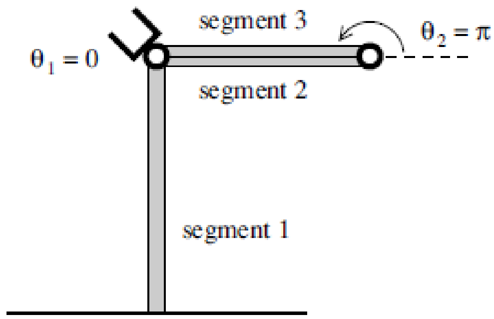


FIGURE 5. At a kinematic singularity

Also close to kinematic singularities problems can occur, since the Jacobian matrix will almost have non-maximal rank and in this situation very large joint space velocity may be needed to archive a small configuration space velocity.

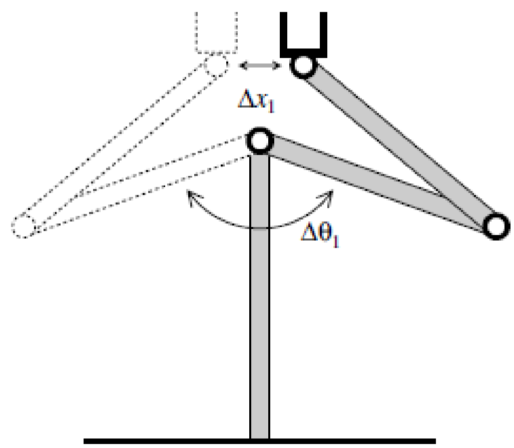


FIGURE 6. Near a kinematic singularity.

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[1] D. Cox, J. Little, D. O'Shea *Ideals, Varieties, and Algorithms,* 2nd edition. Springer, New York, 1997.
[2] T. Becker, V. Weispfenning *Gröbner Bases.* Springer, New York, 1993.