Exercise sheet 9

meeting on **28/05/2019**

To simplify computations you may use a Computer Algebra System.

Exercise 34 Let $\mathcal{C}$ be an affine curve defined by the polynomial $f \in K[x,y]$ of degree d, $P = (a,b) \in \mathbb{A}^2(K)$ be such that $mult_P(\mathcal{C}) = r \geq 1$ and $T(x,y) = (x + a_0, y + b_0)$ be a linear change of coordinates. Show the following:

a) There is choice for a_0, b_0 such that T applied to the first non-vanishing term of the Taylor expansion of f transforms it into a homogeneous polynomial of degree r.

b) For every $a_0, b_0 \in K$ and $f^T(x,y) := f(x + a_0, y + b_0)$ it holds that $mult_{T^{-1}(P)}(f^T) = r$.

c) If $a = b = 0$, then f has the form $f = f_r + f_{r+1} + \cdots + f_d$, where f_j are forms of degree j. Moreover, the tangents to $\mathcal{C}$ at P are the linear factors of r.

To conclude, $mult_P(\mathcal{C})$ can be computed by moving P to the origin and then reading off the lowest occurring degree.

Exercise 35 Compute and classify all (including projective) singularities of the curves defined by the following polynomials.

a) $f_1 = x^4 + y^4 - x^2y^2$.

b) $f_2 = y^2 + (x^2 - 5)(4x^4 - 20x^2 + 25)$.

c) $f_3 = x^4 + x^2(2 + y^2) - 2x^3y - 6xy + 1$.

Exercise 36 Compute the intersection points and its multiplicity of the curves given by the polynomials $f = (x^2 + y^2)^3 - 4x^2y^2$ and $g = x^2 + y^2 - 1$. Verify your result by Bézout's Theorem.

Exercise 37 Find a polynomial defining an irreducible cubic curve with one singularity, namely a non-ordinary double point at the origin with tangent $x + y$, and $(1 : 1 : 0)$ and $(0 : 1 : 0)$ as curve points at infinity.