Exercise sheet 8

meeting on **21/05/2019**

Exercise 30 Let $V = V(z^2 - (x^2 + y^2 - 1)(4 - x^2 - y^2)) \subset \mathbb{R}^3$ and let $\pi : V \to \mathbb{R}^2$ be the projection $\pi(x, y, z) := (x, y)$.

a) For which subsets $R \subset \mathbb{R}^2$ does $(a, b) \in R$ imply $\pi^{-1}(a, b)$ consists of two points, one point, no point?

b) Conclude whether π can be a regular isomorphism. Can you make π to a regular isomorphism by restricting the domain of definition to a subvariety $W \subset V$ and the image correspondingly?

c) Visualize V and $\pi(V)$.

Exercise 31 Show that if $V \subset \mathbb{A}^n$ is a reducible algebraic set, then $K[x_1, \ldots, x_n]/I(V)$ is in general not an integral domain by considering the following example.

a) Let $V = V(xy, xz) \subset \mathbb{R}^3$ and $f = y^2 + z^3, g = x^2 - x$. Show that neither f nor g but their product $f \cdot g$ is identically zero on V.

b) Let $V_1 = V \cap V(f), V_2 = V \cap V(g)$. Show that $V = V_1 \cup V_2$.

Exercise 32 a) Let $V = V(x^2 + 1) \subset \mathbb{R}[x]$. Compute $\Gamma(V)$.

b) Show that $\mathbb{R}[x]/\langle x^2 - 4x + 3\rangle$ is not an integral domain. Conclude that $\langle x^2 - 4x + 3\rangle$ is reducible.

Exercise 33 Let $V = V(y^5 - x^2) \subset \mathbb{R}^2$.

a) Show that every element in $\Gamma[V]$ can be uniquely written as $a(y) + xb(y)$ with $a, b \in \mathbb{R}[y]$.

b) Conclude that V and $\mathbb{R}$ cannot be isomorphic as a variety by showing that there is no ring isomorphism between $\Gamma[V]$ and $\mathbb{R}[t]$.

c) Show that $\mathbb{R}(V)$ can be identified as $\{a(y) + xb(y) \mid a, b \in \mathbb{R}(y)\}$.