Exercise sheet 7

meeting on **14/05/2019**

Exercise 26 [Theorem 5.2.1] Let $f, g \in R[x_1, \ldots, x_n]$, F, G forms in $R[x_1, \ldots, x_n, x_{n+1}]$. Then show the following:

a) $(F \cdot G)_* = F_* \cdot G_*$ and $(f \cdot g)^* = f^* \cdot g^*$.

b) If r is the highest power such that $x_{n+1}^r \mid F$, then $x_{n+1}^r \cdot (F_*)^* = F$, whereas $(f^*)_* = f$.

c) $(F+G)_* = F_* + G_*$ and $x_{n+1}^t (f+g)^* = x_{n+1}^{\max(0,r-s)} f^* + x_{n+1}^{\max(0,s-r)} g^*$, where $r = \deg(g), s = \deg(f), t = \max(r,s) - \deg(f+g)$.

Exercise 27 Consider the affine variety $V = V(y - x^3 + x)$.

a) Compute the projective closure V^* of V.

b) Dehomogenize V^* one time with respect to x and one time with respect to y and visualize the resulting affine varieties to get a better understanding of V^*.

Exercise 28 a) Show that every homogenous polynomial $f \in \mathbb{C}[x, y]$ of degree d can be factored into linear homogenous polynomials, i.e.

$$f(x, y) = \prod_{i=1}^{d} (a_i x + b_i y).$$

b) Let $g(x) = x^3 - 2x^2 - x + 2$. Compute the homogenization g^* and a decomposition of $V_p(g)$. Visualize $V_a(g)$, $V_p(g)$ and the cone $C(V_p(g))$.

Exercise 29 Let $I_0 = \langle x_1, \ldots, x_{n+1} \rangle \subset K[x_1, \ldots, x_{n+1}]$.

a) Show that every homogenous ideal $\emptyset \neq I \subset K[x_1, \ldots, x_{n+1}]$ is contained in I_0.

b) Compute for any $r \in \mathbb{N}$ the ideal I_0^r and an appropriate set of generators. Deduce that every form of degree greater or equal to r is in I_0^r.

c) Consider $V_p(I_0)$ and $V_a(I_0)$ and show that $I_p(V_p(I_0)) \neq I_a(V_a(I_0))$. Compare to Lemma 5.2.5.