# Exercise sheet 6

meeting on **30/04/2019**

---

**Exercise 22** Let $I$ be an ideal and $J$ be a primary ideal of $R$, where $R$ is a commutative ring with 1.

a) Let $I$ be primary and $\sqrt{I} = \sqrt{J}$. Show that then $I \cap J$ is primary.

b) Show that if $I$ is prime and $J \subseteq I$, then $\sqrt{J} \subseteq I$.

**Exercise 23** Let $I$ be a primary ideal with $\sqrt{I} = P$ and $f \in k[x_1, \ldots, x_n]$. Show the following:

a) If $f \in I$, then $I : \langle f \rangle = \langle 1 \rangle$;

b) If $f \notin I$, then $I : \langle f \rangle$ is primary with $\sqrt{I : \langle f \rangle} = P$;

c) If $f \notin P$, then $I : \langle f \rangle = I$.

**Exercise 24** Let $f = x^2y - z^2$ and $I = \langle xz - y^2, x^3 - yz \rangle$.

a) Compute $I : \langle f \rangle$ and show that it is prime.

b) Show that $I$ has the decomposition $\langle x, y \rangle \cap \langle xz - y^2, x^3 - yz, x^2y - z^2 \rangle$.

c) Show that also $\langle xz - y^2, x^3 - yz, x^2y - z^2 \rangle$ is prime by computing a (polynomial) parametrization of the corresponding variety.

Based on the previous two examples it is worth mentioning that for every radical ideal $I \subset k[x_1, \ldots, x_n]$ with prime decomposition

$$I = \bigcap_{i=1}^{r} P_i,$$

the $P_i$'s can be found in the set $\{I : \langle f \rangle \mid f \in k[x_1, \ldots, x_n]\}$. You may think about what this means for the algebraic set $V(I)$ and its decomposition.

**Exercise 25** a) Consider the cuspidal cubic given by $y^2 = ax^3$ with $a \neq 0$. Compute the intersection points of two distinct cuspidal curves.

b) The relations $y - x^2 = 0,\ z - x^3 = 0$ define a space curve in $\mathbb{R}^3$. Consider the corresponding algebraic set projectively and compute its points at infinity. Use this to find two proper (projective) subvarieties.