Exercise sheet 5

meeting on **09/04/2019**

Exercise 18 a) Let I be an ideal of $\mathbb{C}[x_1, \dots, x_n]$ such that $V(I)$ is finite. Show that for every $1 \leq k \leq n-1$ and $(a_{k+1}, \dots, a_n) \in V(I_k)$ there exists an $a_k \in \mathbb{C}$ such that $(a_k, a_{k+1}, \dots, a_n) \in V(I_{k-1})$, where $I_0 = I$.

b) Let I be an ideal of $\mathbb{C}[x_1, \dots, x_n]$ and $V(I_k) \neq \emptyset$ for an $1 \leq k \leq n-1$. Then $V(I_{k-1}) \neq \emptyset$.

Exercise 19 Let $f, g \in \mathbb{C}[x_1, \dots, x_n]$ be irreducible such that f and g have the same roots. Show that f is equal to g up to multiplication with a scalar.

Exercise 20 Consider the system of equations

$$x^2 + 2y^2 = 3, \; x^2 + xy + y^2 = 3.$$

a) If I is the ideal generated by these equations, find generators for $I \cap K[x]$ and $I \cap K[y]$.

b) Compute $V(I)$.

c) Which elements in $V(I)$ are rational?

d) Repeat the previous tasks with the equations $x^2 + 2y^2 = 2, \; x^2 + xy + y^2 = 2$.

Exercise 21 In the proof of Theorem 4.2.6 the following is needed:
Let $P_1, \dots, P_r \in \mathbb{A}^n(K)$ be pairwise different. Construct polynomials $f_1, \dots, f_r \in K[x_1, \dots, x_n]$ such that

$$f_j(P_i) = \begin{cases} 0, i \neq j \\ 1, i = j \end{cases} .$$