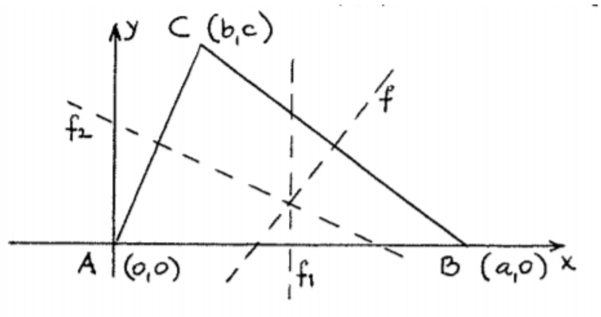
Exercise sheet 2

meeting on **19/03/2019**

Exercise 6 A well-known theorem in geometry is the following: Every triangle ABC in $\mathbb{R}^2$ the lines orthogonal to the sides of the triangle, going through the midpoint of the corresponding side, have a point in common.



We want to prove this theorem by using Groebner bases:

a) Write f_1, f_2, f as polynomials in $\mathbb{R}[x, y]$ depending on the parameters a, b, c.

b) Show that $f \in \sqrt{\langle f_1, f_2 \rangle}$.

Exercise 7 [Lemma 2.2.8] Let I be an ideal in $K[x_1, \ldots, x_n]$ and let $p = (x_1 - a_1) \cdots (x_1 - a_d)$, where $a_1, \ldots, a_d$ are distinct. We want to show that

$$I + \langle p \rangle = \bigcap_{j=1}^{d} (I + \langle x_1 - a_j \rangle).$$

a) First show that $I + \langle p \rangle \subseteq \bigcap_j (I + \langle x_1 - a_j \rangle)$.

b) For the converse direction let $p_j = \prod_{i \neq j} (x_1 - a_i)$. Prove that $p_j \cdot (I + \langle x_1 - a_j \rangle) \subseteq I + \langle p \rangle$.

c) $p_1, \ldots, p_d$ are relatively prime, i.e. there are $h_1, \ldots, h_d$ such that $\sum_{j=1}^{d} h_j p_j = 1$. Show with this property and (b) that $\bigcap_{j=1}^{d} (I + \langle x_1 - a_j \rangle) \subseteq I + \langle p \rangle$.

Exercise 8 Let $I_1 = \langle x^3 + y \rangle, I_2 = \langle x - y^3, xy^2 \rangle$ and $I_3 = \langle y^3(1 - x) + x(x - 1) \rangle$.

a) Compute a basis for $I_4 = I_1 \cdot (I_2 \cap I_3), I_5 = I_1 + (I_2 \cap I_3)$ and $I_6 = I_4 : I_5$.

b) Visualize the corresponding algebraic sets of (a).

c) Check that $I_7 = I_6 + \langle x - (y + 1)^2 \rangle$ is zero-dimensional and compute $\sqrt{I_7}$.

Exercise 9 a) Compute a basis for $Syz(\{f_1, f_2\})$ with $f_1, f_2 \in K[x]$.

i) Does a non-trivial second syzygy exist?

ii) Can we generalize this to the case where $f_1, f_2 \in K[x, y]$?

b) Calculate for general $F = \{f_1, \ldots, f_m\} \subset K[x_1, \ldots, x_n]$ some (first) syzygies of F.