# Exercise sheet 1

meeting on **12/03/2019**

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**Exercise 1** Let $k$ be a field.

a) Let $f \in k[x_1, \ldots, x_n]$ such that $f \notin \langle x_1, \ldots, x_n \rangle$. Show that

$$\langle x_1, \ldots, x_n, f \rangle = k[x_1, \ldots, x_n].$$

b) Let $I \trianglelefteq k[x_1, \ldots, x_n]$ be a principal ideal (it is generated by one element $f \in k[x_1, \ldots, x_n]$). Show that any finite set $F \subset I$ containing $f$ is a Groebner bases for $I$.

**Exercise 2** Let $I \trianglelefteq k[x_1, \ldots, x_n]$ be an ideal and $G$ a Groebner bases of $I$. Let $f, g \in k[x_1, \ldots, x_n]$ and denote the normal form of $f$ modulo $G$ by $\underline{f}$. Then the following hold:

a) $\underline{f} = \underline{g}$ if and only if $f - g \in I$.

b) $\underline{f + g} = \underline{f} + \underline{g}$.

**Exercise 3** Let

$$f_1 = y^2 - xy - xz + 1,$$
$$f_2 = 2xy - 2xz + x^2 + x,$$
$$f_3 = 2xy - 2xz + x^2 + 1.$$

Compute the common roots of $f_1, f_2, f_3$ both by resultants and by Groebner bases.

**Exercise 4** Let

$$a = a_0 + a_1 x + a_2 x^2, \ b = b_0 + b_1 x$$

be polynomials in $\mathbb{Q}[x]$ with unknown coefficients. Compute the polynomial relation between these coefficients if $a$ and $b$ have a common factor.

**Exercise 5** Prove or disprove that the following rings are Noetherian:

a) Principal ideal rings such as the ring of integers $\mathbb{Z}$.

b) The Ring of polynomials in infinitely many variables such as $\mathbb{Q}[x_1, x_2, \ldots]$.

c) The ring of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.