Rewriting

Part 2. Terms, Substitutions, Identities, Semantics

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Purpose of the Lecture

- Introduce syntactic notions:
  - Terms
  - Substitutions
  - Identities
- Define semantics.
- Establish connections between syntax and semantics.

Syntax

Semantics

Syntax

- Alphabet
- Terms

Alphabet

A first-order alphabet consists of the following sets of symbols:

- A countable set of variables $\mathcal{V}$.
- For each $n \geq 0$, a set of n-ary function symbols $\mathcal{F}^n$.
- Elements of $\mathcal{F}^0$ are called constants.
- Signature: $\mathcal{F} = \cup_{n \geq 0} \mathcal{F}^n$.
- $\mathcal{V} \cap \mathcal{F} = \emptyset$.

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Notation:

- x, y, z for variables.
- f, g for function symbols.
- a, b, c for constants.

Terms

Definition 2.1

The set of terms $T(\mathcal{F}, \mathcal{V})$ over $\mathcal{F}$ and $\mathcal{V}$:

- $\mathcal{V} \subseteq T(\mathcal{F}, \mathcal{V})$ (every variable is a term).
- For all $t_1, \ldots, t_n \in T(\mathcal{F}, \mathcal{V})$ and $f \in \mathcal{F}^n$ and $n \geq 0$, we have $f(t_1, \ldots, t_n) \in T(\mathcal{F}, \mathcal{V})$
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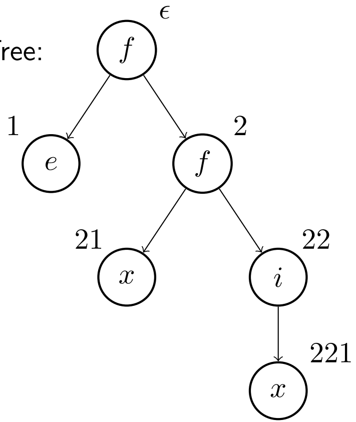
Example:

- $e \in \mathcal{F}^0$, $i \in \mathcal{F}^1$, $f \in \mathcal{F}^2$.
- $f(e, f(x, i(x))) \in T(\mathcal{F}, \mathcal{V})$.

Tree Representation of Terms

Term: $f(e, f(x, i(x)))$

Tree:



Positions: $\epsilon, 1, 2, 21, 22, 221$

Positions

Definition 2.2

Let $t \in T(\mathcal{F}, \mathcal{V})$. The set of positions of t, $\mathcal{P}os(t)$, is a set of strings of positive integers, defined as follows:

- If $t = x$, then $\mathcal{P}os(t) := \{\epsilon\}$,
- If $t = f(t_1, \ldots, t_n)$, then

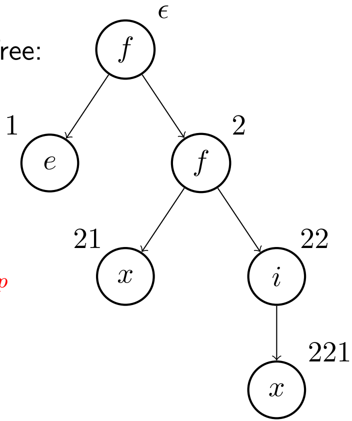
$$\mathcal{P}os(t) := \{\epsilon\} \cup \{ip \mid 1 \leq i \leq n,\ p \in \mathcal{P}os(t_i)\}.$$

- Prefix ordering on positions: $p \leq q$ iff $pp' = q$ for some p'.

More Notions about Terms

Term: $t = f(e, f(x, i(x)))$

Tree:

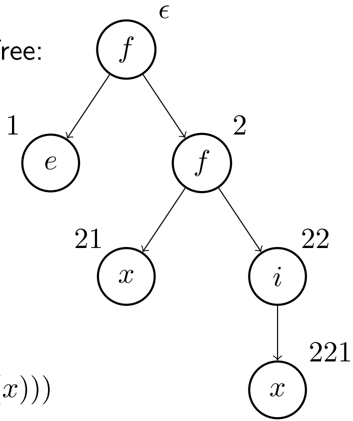


Subterm of t at position p: $t|_p$

$$
\begin{aligned}
t|_2 &= f(x, i(x)) \\
t|_{21} &= x \\
t|_{22} &= i(x)
\end{aligned}
$$

More Notions about Terms

Term: $t = f(e, f(x, i(x)))$ Tree:



Replacing a subterm
at position p by s: $t[s]_p$

$$\begin{aligned} t[a]_\epsilon &= a \\ t[g(a,a)]_{21} &= f(e, f(g(a,a), i(x))) \\ t[i(y)]_{22} &= f(e, f(x, i(y))) \end{aligned}$$

More Notions about Terms

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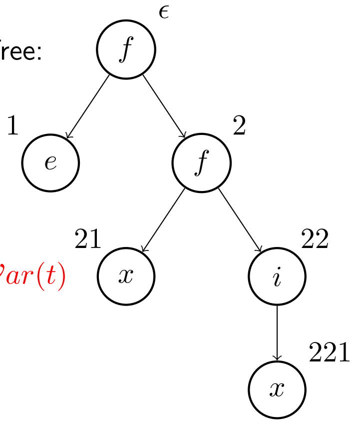

A set of variables occurring in t: $\mathcal{V}ar(t)$

$$
\begin{array}{lcl}
\mathcal{V}ar(t) & = & \{x\} \\
\mathcal{V}ar(t[a]_2) & = & \emptyset \\
\mathcal{V}ar(t|_{22}) & = & \{x\}
\end{array}
$$

More Notions about Terms

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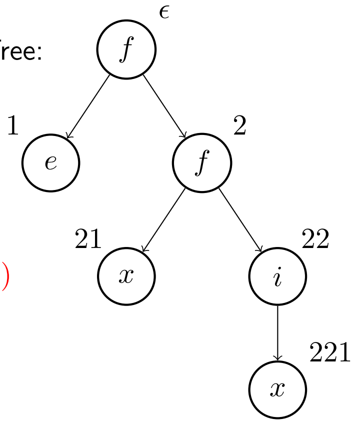

A size of t: $|t| = card(\mathcal{P}os(t))$

$$\begin{array}{lcl} |t| & = & 6 \\ |t[a]_2| & = & 3 \\ |t|_{22}| & = & 2 \end{array}$$

More Notions about Terms

- Ground term: A term without occurrences of variables.
- Ground t: $\mathcal{V}ar(t) = \emptyset$.
- $T(\mathcal{F})$: The set of all ground terms over $\mathcal{F}$.

Substitutions

- A $T(\mathcal{F}, \mathcal{V})$-substitution: A function $\sigma : \mathcal{V} \to T(\mathcal{F}, \mathcal{V})$, whose domain

$$\mathcal{D}om(\sigma) := \{x \mid \sigma(x) \neq x\}$$

is finite.

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$$\mathcal{VR}an(\sigma) := \mathcal{V}ar(\mathcal{R}an(\sigma)).$$

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- Notation: lower case Greek letters $\sigma, \vartheta, \varphi, \psi, \ldots$.
  Identity substitution: ε.

Substitutions

- Notation: If $\mathcal{D}om(\sigma) = \{x_1, \ldots, x_n\}$, then σ can be written as the set

$$\{x_1 \mapsto \sigma(x_1), \ldots, x_n \mapsto \sigma(x_n)\}.$$

- Example:

$$\{x \mapsto i(y), y \mapsto e\}.$$

Substitutions

- The substitution σ can be extended to a mapping

$$\sigma : T(\mathcal{F}, \mathcal{V}) \to T(\mathcal{F}, \mathcal{V})$$

  by induction:

$$\sigma(f(t_1, \ldots, t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n)).$$

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- Example:

$$\begin{aligned} \sigma &= \{x \mapsto i(y), y \mapsto e\}. \\ t &= f(y, f(x, y)) \\ \sigma(t) &= f(e, f(i(y), e)) \end{aligned}$$

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- $\mathcal{S}ub$: The set of substitutions.

More Notions about Substitutions

- Composition of ϑ and σ:

$$\sigma\vartheta(x) := \sigma(\vartheta(x)).$$

- Composition of two substitutions is again a substitution.
- Composition is associative but not commutative.

More Notions about Substitutions

Algorithm for obtaining a set representation of a composition of two substitutions in a set form.

- Given:

$$\theta = \{x_1 \mapsto t_1, \dots, x_n \mapsto t_n\}$$
$$\sigma = \{y_1 \mapsto s_1, \dots, y_m \mapsto s_m\},$$

the set representation of their composition $\sigma\theta$ is obtained from the set

$$\{x_1 \mapsto \sigma(t_1), \dots, x_n \mapsto \sigma(t_n), y_1 \mapsto s_1, \dots, y_m \mapsto s_m\}$$

by deleting

- all $y_i \mapsto s_i$'s with $y_i \in \{x_1, \dots, x_n\}$,
- all $x_i \mapsto \sigma(t_i)$'s with $x_i = \sigma(t_i)$.

More Notions about Substitutions

Example 2.1 (Composition)

$$\begin{aligned} \theta &= \{x \mapsto f(y), y \mapsto z\}. \\ \sigma &= \{x \mapsto a, y \mapsto b, z \mapsto y\}. \\ \sigma\theta &= \{x \mapsto f(b), z \mapsto y\}. \end{aligned}$$

More Notions about Substitutions

- t is an instance of s iff there exists a σ such that

$$\sigma(s) = t.$$

- Notation: $t \gtrsim s$ (or $s \lesssim t$).
- Reads: t is more specific than s, or s is more general than t.
- $\gtrsim$ is a quasi-order.
- Strict part: $>$.

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- Notation: $t \gtrsim s$ (or $s \lesssim t$).
- Reads: t is more specific than s, or s is more general than t.
- $\gtrsim$ is a quasi-order.
- Strict part: $>$.
- Example: $f(e, f(i(y), e)) \gtrsim f(y, f(x, y))$, because

$$\sigma(f(y, f(x, y))) = f(e, f(i(y), e)$$

for $\sigma = \{x \mapsto i(y), y \mapsto e\}$

Identities

- An identity over $T(\mathcal{F}, \mathcal{V})$: a pair $(s, t) \in T(\mathcal{F}, \mathcal{V}) \times T(\mathcal{F}, \mathcal{V})$.
- Written: $s \approx t$.
- s – left hand side, t – right hand side.

Identities

- Given a set E of identities.
- The reduction relation $\rightarrow_E \subseteq T(\mathcal{F},\mathcal{V}) \times T(\mathcal{F},\mathcal{V})$:

  $s \rightarrow_E t$ iff

  there exist $(l, r) \in E$, $p \in \mathcal{P}os(s)$, $\sigma \in \mathcal{S}ub$

  such that $s|_p = \sigma(l)$ and $t = s[\sigma(r)]_p$

- Sometimes written $s \rightarrow^p_E t$.

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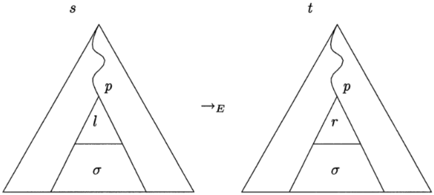

Identities

Example 2.2

- Let G be the set of identities consisting of
  (1) $f(x, f(y, z)) \approx f(f(x, y), z)$
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$$f(i(e), f(e, e))$$

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$$\begin{array}{ll} & f(i(e), f(e, e)) \\ \to_G^{\epsilon} & f(f(i(e), e), e) \quad [(1),\ \sigma_1 = \{x \mapsto i(e), y \mapsto e, z \mapsto e\}] \end{array}$$

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\to_G^{1} & f(e, e) & [(3),\ \sigma_2 = \{x \mapsto e\}] \\
\to_G^{\epsilon} & e & [(2),\ \sigma_3 = \{x \mapsto e\}]
\end{array}
$$

Identities

- $\xrightarrow{*}_E$: Reflexive transitive closure of $\rightarrow_E$.
- $\xleftrightarrow{*}_E$: Reflexive transitive symmetric closure of $\rightarrow_E$.
- An important problem of equational reasoning:
  Design decision procedures for $\xleftrightarrow{*}_E$.

Characterizations of $\overset{*}{\longleftrightarrow}_E$

- Syntactic characterization
- Semantic characterization.

Syntactic characterization of $\stackrel{*}{\longleftrightarrow}_E$

$\equiv$: A binary relation on $T(\mathcal{F}, \mathcal{V})$.

- $\equiv$ is closed under substitutions iff

  $s \equiv t$ implies $\sigma(s) \equiv \sigma(t)$ for all s, t, σ.

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  $s_1 \equiv t_1, \ldots, s_n \equiv t_n$ imply $f(s_1, \ldots, s_n) \equiv f(t_1, \ldots, t_n)$
  for all $s_1, \ldots, s_n, t_1, \ldots, s_n$, $n \geq 0$, $f \in \mathcal{F}^n$.

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  for all $s_1, \ldots, s_n, t_1, \ldots, s_n$, $n \geq 0$, $f \in \mathcal{F}^n$.
- $\equiv$ is compatible with $\mathcal{F}$-operations iff $s \equiv t$ implies
  $f(s_1, \ldots, s_{i-1}, s, s_{i+1}, \ldots, s_n) \equiv$
  $f(s_1, \ldots, s_{i-1}, t, s_{i+1}, \ldots, s_n)$ for all
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  $s_1, \ldots, s_{i-1}, s, t, s_{i+1}, \ldots, s_n \in T(\mathcal{F}, \mathcal{V})$, $n \geq 0$, $f \in \mathcal{F}^n$.
- $\equiv$ is compatible with $\mathcal{F}$-contexts iff $s \equiv t$ implies $r[s]_p \equiv r[t]_p$
  for all $\mathcal{F}$-terms r and positions $p \in \mathcal{P}os(r)$.

Syntactic characterization of $\overset{*}{\longleftrightarrow}_E$

Lemma 2.1

Let E be a set of $\mathcal{F}$-identities. Then $\rightarrow_E$ is closed under substitutions and compatible with $\mathcal{F}$-operations.

Proof.

Follows from the definition of $\rightarrow_E$. □

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Lemma 2.2

Let $\equiv$ be a binary relation on $T(\mathcal{F}, \mathcal{V})$. Then $\equiv$ is compatible with $\mathcal{F}$-operations iff it is compatible with $\mathcal{F}$-contexts.

Proof.

The ($\Rightarrow$) direction can be proved by induction on the length of the position p in the context. The ($\Leftarrow$) direction is obvious. □

Syntactic characterization of $\stackrel{*}{\longleftrightarrow}_E$

Exercise: Which of the following relations is closed under substitutions, closed under $\mathcal{F}$-operations, or compatible with $\mathcal{F}$-operations?

- $s \equiv t$ iff t is a subterm of s.
- $s \equiv t$ iff t is an instance of s.
- $s \equiv t$ iff $\mathcal{V}ar(s) \subseteq \mathcal{V}ar(t)$.

Syntactic characterization of $\stackrel{*}{\longleftrightarrow}_E$

Lemma 2.3

Let $\equiv$ be a binary relation on $T(\mathcal{F}, \mathcal{V})$. If $\equiv$ is reflexive and transitive, then it is compatible with $\mathcal{F}$-operations iff it is closed under $\mathcal{F}$-operations.

Syntactic characterization of $\overset{*}{\longleftrightarrow}_E$

Lemma 2.3

Let $\equiv$ be a binary relation on $T(\mathcal{F},\mathcal{V})$. If $\equiv$ is reflexive and transitive, then it is compatible with $\mathcal{F}$-operations iff it is closed under $\mathcal{F}$-operations.

Proof.

($\Rightarrow$) Assume $s_i \equiv t_i$ for all $1 \leq i \leq n$. By compatibility we have

$$
\begin{aligned}
f(s_1, s_2, \ldots, s_n) &\equiv f(t_1, s_2, \ldots, s_n)\\
f(t_1, s_2, \ldots, s_n) &\equiv f(t_1, t_2, \ldots, s_n)\\
&\ldots\\
f(t_1, t_2, \ldots, s_n) &\equiv f(t_1, t_2, \ldots, t_n)
\end{aligned}
$$

Transitivity of $\equiv$ implies $f(s_1, \ldots, s_n) \equiv f(t_1, \ldots, t_n)$.

($\Leftarrow$) Using reflexivity of $\equiv$.

Syntactic characterization of $\overset{*}{\longleftrightarrow}_E$

Theorem 2.1

Let E be a set of identities. $\overset{*}{\longleftrightarrow}_E$ is the smallest equivalence relation on $T(\mathcal{F}, \mathcal{V})$ that

(a) *contains E,*

(b) *is closed under substitutions, and*

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$\overset{*}{\longleftrightarrow}_E$ is an equivalence relation by definition.

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(a) Obvious.

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Proof (Cont.)

(b) Assume $s \overset{*}{\longleftrightarrow}_E t$. Prove $\sigma(s) \overset{*}{\longleftrightarrow}_E \sigma(t)$ for a σ by induction on the length of $\overset{*}{\longleftrightarrow}_E$ chain.

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Proof (Cont.)

(b) Assume $s \overset{*}{\longleftrightarrow}_E t$. Prove $\sigma(s) \overset{*}{\longleftrightarrow}_E \sigma(t)$ for a σ by induction on the length of $\overset{*}{\longleftrightarrow}_E$ chain. IB $s = t$: Obvious. IH for $s \overset{*}{\longleftrightarrow}_E t$. IS: Let $s \overset{*}{\longleftrightarrow}_E t \leftrightarrow_E t'$. By case distinction on $\leftrightarrow_E$.

Syntactic characterization of $\overset{*}{\leftrightarrow}_E$

Theorem 2.1

Let E be a set of identities. $\overset{*}{\leftrightarrow}_E$ is the smallest equivalence relation on $T(\mathcal{F}, \mathcal{V})$ that

(a) *contains E,*

(b) *is closed under substitutions, and*

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- $t \to_E t'$: By IH: $\sigma(s) \overset{*}{\leftrightarrow}_E \sigma(t)$.
  $t \to_E t' \Rightarrow \sigma(t) \to_E \sigma(t') \Rightarrow \sigma(t) \overset{*}{\leftrightarrow}_E \sigma(t')$.
  By transitivity of $\overset{*}{\leftrightarrow}_E$: $\sigma(s) \overset{*}{\leftrightarrow}_E \sigma(t')$.

Syntactic characterization of $\xleftrightarrow{*}_E$

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Let E be a set of identities. $\xleftrightarrow{*}_E$ is the smallest equivalence relation on $T(\mathcal{F}, \mathcal{V})$ that

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Proof (Cont.)

(b) Assume $s \xleftrightarrow{*}_E t$. Prove $\sigma(s) \xleftrightarrow{*}_E \sigma(t)$ for a σ by induction on the length of $\xleftrightarrow{*}_E$ chain. IB $s = t$: Obvious. IH for $s \xleftrightarrow{*}_E t$. IS: Let $s \xleftrightarrow{*}_E t \leftrightarrow_E t'$. By case distinction on $\leftrightarrow_E$.

- $t \to_E t'$: By IH: $\sigma(s) \xleftrightarrow{*}_E \sigma(t)$.
  $t \to_E t' \Rightarrow \sigma(t) \to_E \sigma(t') \Rightarrow \sigma(t) \xleftrightarrow{*}_E \sigma(t')$.
  By transitivity of $\xleftrightarrow{*}_E$: $\sigma(s) \xleftrightarrow{*}_E \sigma(t')$.
- $t' \to_E t$. Similar to the previous item.

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Let E be a set of identities. $\overset{*}{\longleftrightarrow}_E$ is the smallest equivalence relation on $T(\mathcal{F}, \mathcal{V})$ that

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- $\stackrel{*}{\longleftrightarrow}_E$ is reflexive and transitive and compatible with $\mathcal{F}$-operations (because $\rightarrow_E$ is).

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Proof (Cont.)

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- $\stackrel{*}{\longleftrightarrow}_E$ is reflexive and transitive and compatible with $\mathcal{F}$-operations (because $\rightarrow_E$ is).
- By Lemma 2.3, $\stackrel{*}{\longleftrightarrow}_E$ is closed under $\mathcal{F}$-operations.

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Prove that $\overset{*}{\longleftrightarrow}_E$ is the smallest such relation. Take another equivalence relation $\equiv$ on $T(\mathcal{F}, \mathcal{V})$ which satisfies (a), (b), (c). Prove that $\overset{*}{\longleftrightarrow}_E \subseteq \equiv$.

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- First, prove $\rightarrow_E \; \subseteq \; \equiv$.

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- First, prove $\rightarrow_E \subseteq \equiv$.
- Let $s \rightarrow_E t$. It implies that there exist $(l, r) \in E$, $p \in \mathcal{P}os(s)$, and σ such that $s|_p = \sigma(l)$, $t = s[\sigma(r)]_p$.

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Proof (Cont.)

- $E \subseteq\ \equiv\ \Rightarrow l \equiv r \Rightarrow \sigma(l) \equiv \sigma(r)$.

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Let E be a set of identities. $\overset{*}{\longleftrightarrow}_E$ is the smallest equivalence relation on $T(\mathcal{F}, \mathcal{V})$ that

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Proof (Cont.)

- $E \subseteq \ \equiv \ \Rightarrow l \equiv r \Rightarrow \sigma(l) \equiv \sigma(r)$.
- $\equiv$ is reflexive and transitive and closed under $\mathcal{F}$-operations. By Lemma 2.3, $\equiv$ is compatible with $\mathcal{F}$-operations.

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Let E *be a set of identities.* $\overset{*}{\longleftrightarrow}_E$ *is the smallest equivalence relation on* $T(\mathcal{F}, \mathcal{V})$ *that*

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Proof (Cont.)

- By Lemma 2.2, $\equiv$ is compatible with contexts: $\sigma(l) \equiv \sigma(r)$ implies $u[\sigma(l)]_{pos} \equiv u[\sigma(r)]_{pos}$ for all $u, pos \in \mathcal{P}os(u)$, σ.

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- In particular, $s = s[\sigma(l)]_p \equiv s[\sigma(r)]_p = t$.

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- In particular, $s = s[\sigma(l)]_p \equiv s[\sigma(r)]_p = t$.
- Hence, $s \equiv t$ and $\rightarrow_E \; \subseteq \; \equiv$.

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Proof (Finished).

- $\rightarrow_E \ \subseteq \ \equiv$ implies $\stackrel{*}{\longleftrightarrow} \ \subseteq \ \equiv$, because, by definition, $\stackrel{*}{\longleftrightarrow}$ is the smallest equivalence relation containing $\rightarrow_E$.

Syntactic characterization of $\stackrel{*}{\longleftrightarrow}_E$

Theorem 2.1 says that $\stackrel{*}{\longleftrightarrow}_E$ can be obtained by starting with the binary relation E and closing it under

- reflexivity,
- symmetry,
- transitivity,
- substitutions, and
- $\mathcal{F}$-operations.

describing the closing process leads to equational logic.

Equational Logic

Inference rules:

$$\frac{s \approx t \in E}{E \vdash s \approx t}$$

$$\frac{}{E \vdash s \approx s} \qquad \frac{E \vdash s \approx t}{E \vdash t \approx s} \qquad \frac{E \vdash s \approx t \quad E \vdash t \approx r}{E \vdash s \approx r}$$

$$\frac{E \vdash s \approx t}{E \vdash \sigma(s) \approx \sigma(t)} \qquad \frac{E \vdash s_1 \approx t_1 \quad \cdots \quad E \vdash s_n \approx t_n}{E \vdash f(s_1, \ldots, s_n) \approx f(t_1, \ldots, t_n)}$$

$E \vdash s \approx t$: $s \approx t$ is a syntactic consequence of E, or $s \approx t$ is provable from E.

Equational Logic

Example 2.3

- Let $E = \{a \approx b, f(x) \approx g(x)\}$.
- Prove $E \vdash g(b) \approx f(a)$.

Proof:

$$\dfrac{\dfrac{E \vdash a \approx b}{E \vdash f(a) \approx f(b)}\ \text{(Func. closure)} \qquad \dfrac{E \vdash f(x) \approx g(x)}{E \vdash f(b) \approx g(b)}\ \text{(Subst. inst.)}}{\dfrac{E \vdash f(a) \approx g(b)}{E \vdash g(b) \approx f(a)}\ \text{(Symmetry)}}\ \text{(Transitivity)}$$

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Compare with the derivation of $g(b) \overset{*}{\leftrightarrow}_E f(a)$:

$$g(b) \leftrightarrow_E g(a) \leftrightarrow_E f(a)$$

Convertibility and Provability

Theorem 2.2 (Logicality)

For all E, s, t,

$$s \overset{*}{\leftrightarrow}_E t \ \textit{ iff } \ E \vdash s \approx t.$$

Proof.

Follows from Theorem 2.1. □

Convertibility and Provability

Differences in behavior:

1. The rewriting approach $\overset{*}{\longleftrightarrow}_E$ allows the replacement of a subterm at an arbitrary position in a single step; The inference rule approach $E \vdash$ needs to simulate this with a sequence of small steps.
2. The inference rule approach allows the simultaneous replacement in each argument of an operation; The rewriting approach needs to simulate this by a number of replacement steps in sequence.

Syntax

Semantics

Semantic Algebras

- $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$.
- A is a nonempty set, the carrier.
- $f_{\mathcal{A}} : A^n \to A$ is an interpretation for $f \in \mathcal{F}^n$.

Semantic Algebras

- $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$.
- A is a nonempty set, the carrier.
- $f_{\mathcal{A}} : A^n \to A$ is an interpretation for $f \in \mathcal{F}^n$.

Example 2.4

Two $\{0, s, +\}$-algebras:

$\mathcal{A} = (\mathbb{N}, \{0_{\mathcal{A}}, s_{\mathcal{A}}, +_{\mathcal{A}}\})$ with $0_{\mathcal{A}} = 0$, $s_{\mathcal{A}}(x) = x + 1$, $+_{\mathcal{A}}(x, y) = x + y$.

$\mathcal{B} = (\mathbb{N}, \{0_{\mathcal{B}}, s_{\mathcal{B}}, +_{\mathcal{B}}\})$ with $0_{\mathcal{B}} = 1$, $s_{\mathcal{B}}(x) = x + 1$, $+_{\mathcal{B}}(x, y) = 2x + y$.

Variable Assignment, Interpretation Function

- Variable assignment: $\alpha : \mathcal{V} \to A$
- Interpretation function: $[\alpha]_{\mathcal{A}}(\cdot) : T(\mathcal{F}, \mathcal{V}) \to A$

$$[\alpha]_{\mathcal{A}}(t) = \begin{cases} \alpha(t) & \text{if } t \in \mathcal{V} \\ f_{\mathcal{A}}([\alpha]_{\mathcal{A}}(t_1), \ldots, [\alpha]_{\mathcal{A}}(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

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Example 2.5

$\mathcal{A} = (\mathbb{N}, \{0_{\mathcal{A}}, s_{\mathcal{A}}, +_{\mathcal{A}}\})$ with $0_{\mathcal{A}} = 0$, $s_{\mathcal{A}}(x) = x + 1$, $+_{\mathcal{A}}(x, y) = x + y$.

$\mathcal{B} = (\mathbb{N}, \{0_{\mathcal{B}}, s_{\mathcal{B}}, +_{\mathcal{B}}\})$ with $0_{\mathcal{B}} = 1$, $s_{\mathcal{B}}(x) = x + 1$, $+_{\mathcal{B}}(x, y) = 2x + y$.

$t = s(s(x)) + s(x + y)$, $\alpha(x) = 2$, $\alpha(y) = 3$, $\beta(x) = 1$, $\beta(y) = 4$.

$$[\alpha]_{\mathcal{A}}(t) = 10 \qquad [\beta]_{\mathcal{A}}(t) = 9$$
$$[\alpha]_{\mathcal{B}}(t) = 16 \qquad [\beta]_{\mathcal{B}}(t) = 13$$

Validity, Models

- An equation $s \approx t$ is valid in algebra $\mathcal{A}$, written $\mathcal{A} \models s \approx t$, iff

  $$[\alpha]_{\mathcal{A}}(s) = [\alpha]_{\mathcal{A}}(t)$$

  for all assignments α.
- An $\mathcal{F}$-algebra $\mathcal{A}$ is a model of the set of identities E over $T(\mathcal{F}, \mathcal{V})$ iff $\mathcal{A} \models s \approx t$ for all $s \approx t \in E$.

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Example 2.6

$\mathcal{A} = (\mathbb{N}, \{0_{\mathcal{A}}, s_{\mathcal{A}}, +_{\mathcal{A}}\})$ with $0_{\mathcal{A}} = 0$, $s_{\mathcal{A}}(x) = x + 1$, $+_{\mathcal{A}}(x, y) = x + y$.

$\mathcal{B} = (\mathbb{N}, \{0_{\mathcal{B}}, s_{\mathcal{B}}, +_{\mathcal{B}}\})$ with $0_{\mathcal{B}} = 1$, $s_{\mathcal{B}}(x) = x + 1$, $+_{\mathcal{B}}(x, y) = 2x + y$.

$E = \{0 + y \approx y,\ s(x) + y \approx s(x + y)\}$.

$\mathcal{A}$ is a model of E, while $\mathcal{B}$ is not.

Validity, Models, Equational Theory

- $E \models s \approx t$ iff $s \approx t$ is valid in all models of E.
- $E \models s \approx t$: $s \approx t$ is a semantic consequence of E.
- Equational theory of E:

$$\approx_E := \{(s,t) \mid s,t \in T(\mathcal{F},\mathcal{V}),\ E \models s \approx t\}$$

- Notation: $s \approx_E t$ iff $(s,t) \in\ \approx_E$.

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Example 2.7

- $E = \{0 + y \approx y,\ s(x) + y \approx s(x+y)\}$.

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- $E = \{0 + y \approx y,\ s(x) + y \approx s(x+y)\}$.
- $E \models s(s(0) + s(0)) \approx s(s(s(0)))$.
- $E \not\models x + y \approx y + x$.
- Model $\mathcal{C} = (\mathbb{N}, \{0_\mathcal{C}, s_\mathcal{C}, +_\mathcal{C}\})$ with $0_\mathcal{C} = 0$, $s_\mathcal{C}(x) = x$, $+_\mathcal{C}(x,y) = y$.

Relating Syntax and Semantics

Theorem 2.3 (Birkhoff)

Equational logic is sound and complete:

For all $E, s, t, \quad E \vdash s \approx t \quad$ *iff* $\quad E \models s \approx t.$

Relating Syntax and Semantics

Theorem 2.3 (Birkhoff)

Equational logic is sound and complete:

$$\textit{For all } E, s, t, \quad E \vdash s \approx t \quad \textit{iff} \quad E \models s \approx t.$$

Corollary 2.1

For all $E, s, t,$

$$s \overset{*}{\longleftrightarrow}_E t \quad \textit{iff} \quad E \vdash s \approx t \quad \textit{iff} \quad E \models s \approx t.$$

Validity and Satisfiability

Validity problem:

Given: A set of identities E and terms s and t.

Decide: $s \approx_E t$.

Satisfiability problem:

Given: A set of identities E and terms s and t.

Find: A substitution σ such that $\sigma(s) \approx_E \sigma(t)$.