Rewriting

Part 1. Abstract Reduction

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Literature

- Franz Baader and Tobias Nipkow. Term Rewriting and All That. Cambridge University Press, 1998.
- Book's home page:
  http://www21.in.tum.de/~nipkow/TRaAT/
- Resources about rewriting: http://rewriting.loria.fr/

Motivation

Abstract Reduction Systems

Equational Reasoning

- Restricted class of languages: The only predicate symbol is equality $\approx$.
- Reasoning with equations:
  - derive consequences of given equations,
  - find values for variables that satisfy a given equation.
- At the heart of many problems in mathematics and computer science.

Example: Addition of Natural Numbers

- Equations (identities):

$$x + 0 \approx x$$
$$x + s(y) \approx s(x + y)$$

- How to calculate $s(0) + s(s(0))$?

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$$s(s(s(0)))$$

What is Rewriting

- Process of transforming one expression into another.
- Rules describe how one expression can be rewritten into another.

Identities and Rewriting

- Rewriting as a computational mechanism:
  - Apply given equations in one direction, as rewrite rules.
  - Compute normal forms.
  - Close relationship with functional programming.
  - Example: symbolic differentiation.

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  - Close relationship with functional programming.
  - Example: symbolic differentiation.
- Rewriting as a deduction mechanism:
  - Apply given equations in both directions.
  - Define equivalence classes of terms.
  - Equational reasoning.
  - Example: group theory.

Symbolic Differentiation

- Expressions: Terms built over variables $(u, v, \ldots)$ and the following function symbols:
  - constants $0, 1$ (numbers),
  - constants X, Y (indeterminates),
  - unary symbol D_X (partial derivative with respect to X),
  - binary symbols $+, *$.

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  - $D_X(u * v)$.
  - $(X + Y) * D_X(X * Y)$.

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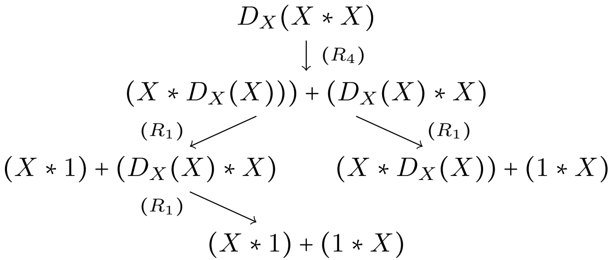

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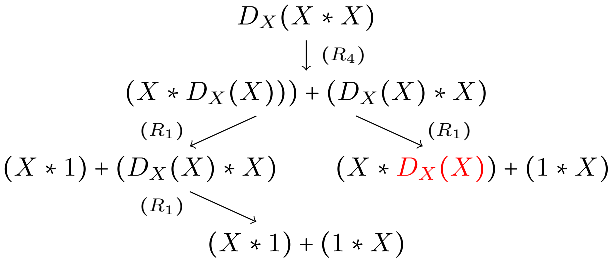

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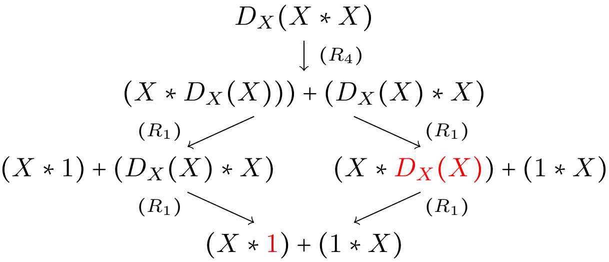

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Properties of Term Rewriting Systems

The symbolic differentiation example can be used to illustrate two most important properties of TRSs:

Properties of Term Rewriting Systems

The symbolic differentiation example can be used to illustrate two most important properties of TRSs:

1. Termination:
   - Is it always the case that after finitely many rule applications we reach an expression to which no more rules apply (normal form)?
   - For symbolic differentiation rules this is the case.
   - But how to prove it?
   - An example of non-terminating rule: $u + v \to v + u$

Properties of Term Rewriting Systems

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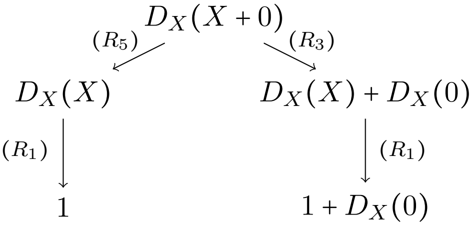
The symbolic differentiation example can be used to illustrate two most important properties of TRSs:

2. Confluence:
   - If there are different ways of applying rules to a given term t, leading to different terms t_1 and t_2, can they be reduced by rule applications to a common term?
   - For symbolic differentiation rules this is the case.
   - But how to prove it?

Properties of Term Rewriting Systems

- Adding the rule $u + 0 \to u$ (R_5) destroys confluence:



- Confluence can be regained by adding $D_X(0) \to 0$ (completion).

Group Theory

- Terms are built over variables and the following function symbols:
  - binary $\circ$,
  - unary i,
  - constant 0.
- Examples of terms:
  - $x \circ (y \circ i(y))$
  - $(0 \circ x) \circ i(0)$
  - $i(x \circ y)$
- Identities (aka group axioms), defining groups:

Associativity of $\circ$	$(x \circ y) \circ z \approx x \circ (y \circ z)$	(G_1)
e left unit	$e \circ x \approx x$	(G_2)
i left inverse	$i(x) \circ x \approx e$	(G_3)

Group Theory

- Identities can be applied in both directions.
- Word problem for identities:
  - Given a set of identities E and two terms s and t.
  - Is it possible to transform s into t, using the identities in E as rewrite rules applied in both directions?
- For instance, is it possible to transform e into $x \circ i(x)$, i.e., is the left inverse also a right-inverse?

Group Theory

$$
\begin{aligned}
(x \circ y) \circ z &\approx x \circ (y \circ z) && (G_1) \\
e \circ x &\approx x && (G_2) \\
i(x) \circ x &\approx e && (G_3)
\end{aligned}
$$

Transform e into $x \circ i(x)$:

$$
\begin{aligned}
e &\approx_{G_3} i(x \circ i(x)) \circ (x \circ i(x)) \\
&\approx_{G_2} i(x \circ i(x)) \circ (x \circ (e \circ i(x))) \\
&\approx_{G_3} i(x \circ i(x)) \circ (x \circ ((i(x) \circ x) \circ i(x))) \\
&\approx_{G_1} i(x \circ i(x)) \circ ((x \circ (i(x) \circ x)) \circ i(x)) \\
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&\approx_{G_1} (i(x \circ i(x)) \circ (x \circ i(x))) \circ (x \circ i(x)) \\
&\approx_{G_3} e \circ (x \circ i(x)) \\
&\approx_{G_2} x \circ i(x)
\end{aligned}
$$

Solving Word Problems by Rewriting?

- Is there a simpler way to solve word problems?
- Try to solve it by rewriting (uni-directional application of identities):

$$s \xrightarrow{*} \hat{s} = \hat{t} \xleftarrow{*} t$$

- Reduce s and t to normal forms $\hat{s}$ and $\hat{t}$.
- Check whether $\hat{s} = \hat{t}$, i.e., syntactically equal. (= is the meta-equality.)

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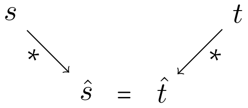

- Reduce s and t to normal forms $\hat{s}$ and $\hat{t}$.
- Check whether $\hat{s} = \hat{t}$, i.e., syntactically equal.
  (= is the meta-equality.)
- But... it would only work if normal forms exist and are unique.

Solving Word Problems by Rewriting?

- In the group theory example, e and $x \circ i(x)$ are equivalent, but it can not be decided by (left-to-right) rewriting: Both terms are in the normal form.
- Uniqueness of normal forms is violated: non-confluence.
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- Normal forms may not exist: The process of reducing a term may lead to an infinite chain of transformations: non-termination.
- Termination and confluence ensure existence and uniqueness of normal forms.
- If a given set of identities leads to non-confluent system, we will try to apply the idea of completion to extend the rewrite system to a confluent one.

Motivation

Abstract Reduction Systems

Abstract vs Concrete

Concrete rewrite formalisms:

- string rewriting
- term rewriting
- graph rewriting
- λ calculus
- etc.

Abstract reduction:

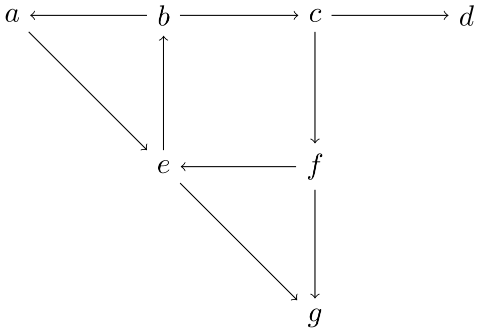
- No structure on objects to be rewritten.
- Abstract treatment of reductions.

Abstract Reduction Systems

- Abstract reduction system (ARS): A pair $(A, \to)$, where
  - A is a set,
  - the reduction $\to$ is a binary relation on A: $\to \subseteq A \times A$.
- Write $a \to b$ for $(a, b) \in \to$.

Abstract Reduction System: Example

- $A = \{a, b, c, d, e, f, g\}$
- $\rightarrow = \left\{ \begin{array}{l} (a,e),(b,a),(b,c),(c,d),(c,f) \\ (e,b),(e,g),(f,e),(f,g) \end{array} \right\}$



Equivalence and Reduction

Again, two views at reductions.

1. Directed computation: Follow the reductions, trying to compute a normal form: $a_0 \to a_1 \to \cdots$
2. View $\to$ as description of $\overset{*}{\leftrightarrow}$.
   - $a \overset{*}{\leftrightarrow} b$ means there is a path between a and b, with arrows traversed in both directions: $a \leftarrow c \to d \leftarrow b$
   - Goal: Decide whether $a \overset{*}{\leftrightarrow} b$.
   - Bidirectional rewriting is expensive.
   - Unidirectional rewriting with subsequent comparison of normal form works if the reduction system is confluent and terminating.

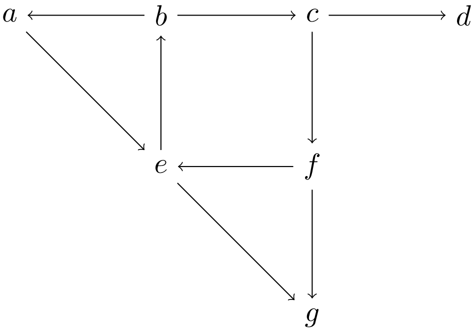
Termination, confluence: central topics.

Basic notions

1. Composition of two relations.
2. Given two relations $R \subseteq A \times B$ and $S \subseteq B \times C$, their composition is defined by

$$R \circ S := \{(x, z) \mid \exists y \in B.\ (x, y) \in R \wedge (y, z) \in S\}$$

Abstract Reduction System: Example



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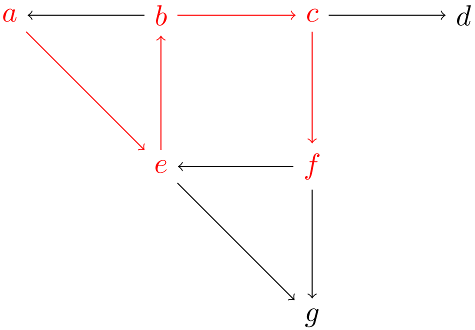

- Finite rewrite sequence: $a \to e \to b \to c \to f$

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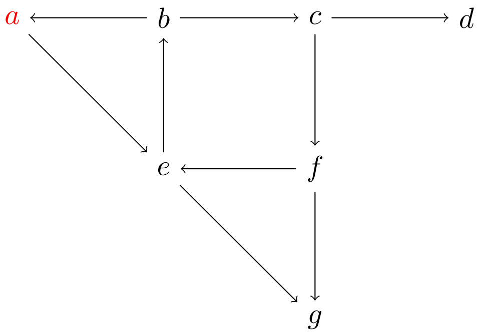

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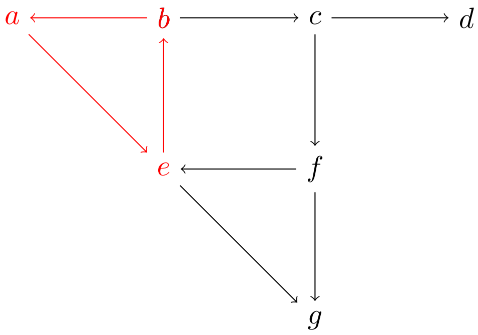

- Finite rewrite sequence: $a \to e \to b \to c \to f$
- Empty rewrite sequence: a
- Infinite rewrite sequence: $a \to e \to b \to a \to \cdots$

Relations Derived from $\to$

$\xrightarrow{0} := \{(x,x) \mid x \in A\}$	identity
$\xrightarrow{i+1} := \xrightarrow{i} \circ \to$	$(i+1)$-fold composition, $i \geq 0$
$\xrightarrow{+} := \cup_{i>0} \xrightarrow{i}$	transitive closure
$\xrightarrow{*} := \xrightarrow{+} \cup \xrightarrow{0}$	reflexive transitive closure
$\xrightarrow{=} := \to \cup \xrightarrow{0}$	reflexive closure
$\xrightarrow{-1} := \{(y,x) \mid (x,y) \in \to\}$	inverse
$\leftarrow := \xrightarrow{-1}$	inverse
$\leftrightarrow := \to \cup \leftarrow$	symmetric closure
$\overset{+}{\leftrightarrow} := (\leftrightarrow)^+$	transitive symmetric closure
$\overset{*}{\leftrightarrow} := (\leftrightarrow)^*$	reflexive transitive symmetric closure

Terminology

- If $x \xrightarrow{*} y$ then we say:
  - x rewrites to y, or
  - there is some finite path from x to y, or
  - y is a reduct of x.

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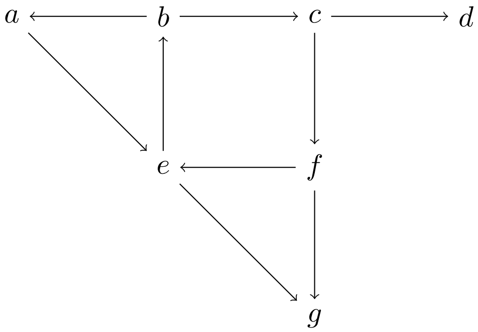

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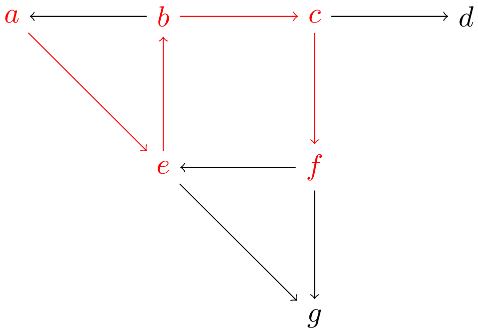

$a \xrightarrow{*} f$

Terminology

- x is reducible iff there exists y such that $x \to y$.

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- x is in normal form (irreducible) iff x is not reducible.

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- y is a normal form of x iff $x \xrightarrow{*} y$ and y is in normal form.

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- If x has a unique normal form, it is denoted by $x\downarrow$.

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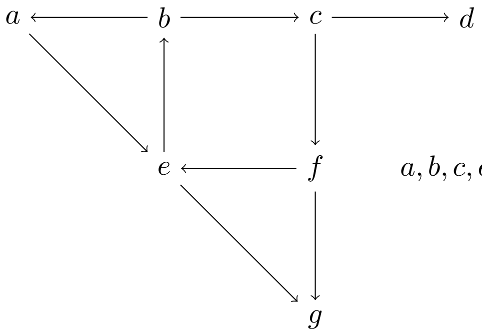

a, b, c, e, f are reducible

Terminology

- x is reducible iff there exists y such that $x \to y$.
- x is in normal form (irreducible) iff x is not reducible.
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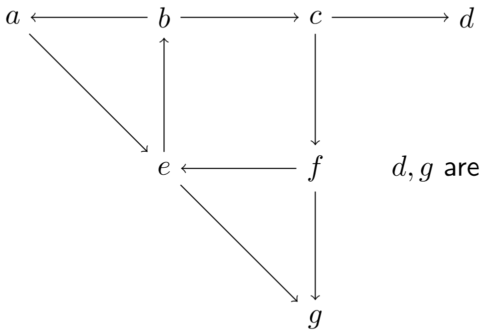

d, g are in a normal form

Terminology

- x is reducible iff there exists y such that $x \to y$.
- x is in normal form (irreducible) iff x is not reducible.
- y is a normal form of x iff $x \xrightarrow{*} y$ and y is in normal form.
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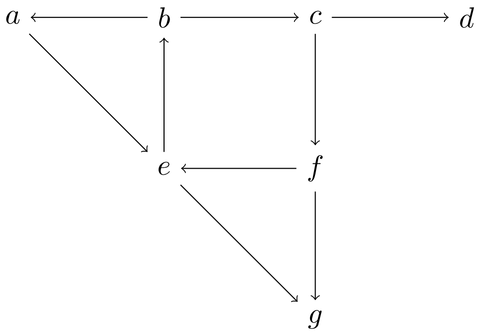

$$b \xrightarrow{!} d$$
$$b \xrightarrow{!} g$$
$$g \xrightarrow{!} g$$

Terminology

- y is direct successor of x iff $x \to y$.

Terminology

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- y is successor of x iff $x \xrightarrow{+} y$.
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- x and y are joinable iff there exists z such that $x \xrightarrow{*} z \xleftarrow{*} y$.

Terminology

- y is direct successor of x iff $x \to y$.
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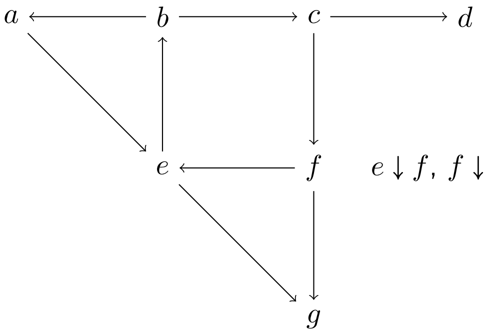

$e \downarrow f$, $f \downarrow d$, $a \downarrow f$, not $g \downarrow d$

Terminology

- y is direct successor of x iff $x \to y$.
- y is successor of x iff $x \xrightarrow{+} y$.
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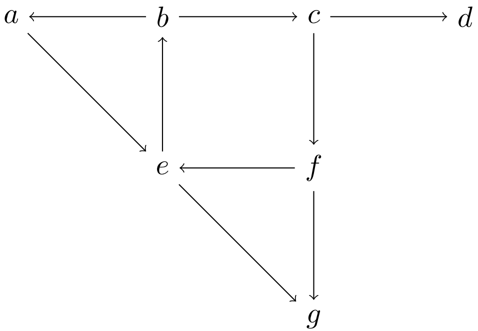

$$g \overset{*}{\leftrightarrow} d$$

Example

1. Let $A := \mathbb{N} - \{0, 1\}$ and $\rightarrow := \{(m, n) \mid m > n$ and n divides $m\}$. Then
   (a) m is in normal form iff m is prime.
   (b) p is a normal form of m iff p is a prime factor of m.
   (c) $m \downarrow n$ iff m and n are not relatively prime.
   (d) $\xrightarrow{+} = \rightarrow$ because $>$ and "divides" are already transitive.
   (e) $\xleftrightarrow{*} = A \times A$.
2. Let $A := \{a, b\}^*$ (the set of words over the alphabet $\{a, b\}$) and $\rightarrow := \{(ubav, uabv) \mid u, v \in A\}$. Then
   (a) w is in normal form iff w is sorted, i.e. of the form a^*b^*.
   (b) Every w has a unique normal form $w\downarrow$, the result of sorting w.
   (c) $w_1 \downarrow w_2$ iff $w_1 \xleftrightarrow{*} w_2$ iff w_1 and w_2 contain the same number of as and bs.

Central Notions

Definition 1.1

A relation $\rightarrow$ is called Church-Rosser (CR) iff

$$x \overset{*}{\leftrightarrow} y \text{ implies } x \downarrow y.$$

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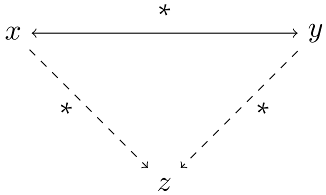
Graphically:



Solid arrows represent universal and dashed arrows existential quantification: $\forall x, y.\ x \overset{*}{\leftrightarrow} y \Rightarrow \exists z.\ x \overset{*}{\to} z \wedge y \overset{*}{\to} z.$

Central Notions: Church-Rosser

Definition 1.2

A relation $\to$ is called confluent (C) iff

$$y_1 \overset{*}{\leftarrow} x \overset{*}{\to} y_2 \text{ implies } y_1 \downarrow y_2.$$

Central Notions: Church-Rosser

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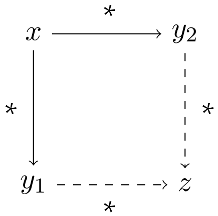

Solid arrows represent universal and dashed arrows existential quantification: $\forall x, y_1, y_2.\ y_1 \overset{*}{\leftarrow} x \overset{*}{\rightarrow} y_2 \Rightarrow \exists z.\ y_1 \overset{*}{\rightarrow} z \overset{*}{\leftarrow} y_2.$

Central Notions: Local Confluence

Definition 1.3
A relation $\to$ is called locally confluent (LC) iff

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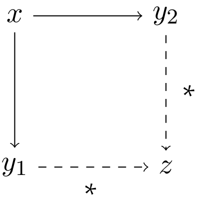

Solid arrows represent universal and dashed arrows existential quantification: $\forall x, y_1, y_2.\ y_1 \leftarrow x \to y_2 \Rightarrow \exists z.\ y_1 \xrightarrow{*} z \xleftarrow{*} y_2$.

Central Notions: T, N, UN, Convergence

Definition 1.4

A relation $\to$ is called

- terminating (T) iff there is no infinite descending chain $a_0 \to a_1 \to \cdots$.
- normalizing (N) iff every element has a normal form.
- uniquely normalizing (UN) iff every element has at most one normal form.
- convergent iff it is both confluent and terminating.

Central Notions: T, N, UN, Convergence

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- terminating (T) iff there is no infinite descending chain $a_0 \to a_1 \to \cdots$.
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- uniquely normalizing (UN) iff every element has at most one normal form.
- convergent iff it is both confluent and terminating.

Alternative terminology:

- Strongly normalizing: terminating.
- Weakly normalizing: normalizing.

Central Notions: CR Reformulated

- Obviously, $x \downarrow y$ implies $x \stackrel{*}{\leftrightarrow} y$.
- Therefore, the Church-Rosser property can be formulated as the equivalence:
- $\rightarrow$ is called Church-Rosser iff

$$x \stackrel{*}{\leftrightarrow} y \text{ iff } x \downarrow y.$$

Properties

1. $T \implies N$

Properties

1. $T \implies N$
2. $T \not\Longleftarrow N$

Properties

1. T $\Longrightarrow$ N
2. T $\not\Longleftarrow$ N

$\circlearrowright\; a \longrightarrow b$

Properties

1. T $\Longrightarrow$ N
2. T $\not\Longleftarrow$ N
3. CR $\Longleftrightarrow$ $\overset{*}{\leftrightarrow} = \downarrow$

$\circlearrowright a \longrightarrow b$

Properties

1. T $\Longrightarrow$ N
2. T $\not\Longleftarrow$ N $\circlearrowright a \longrightarrow b$
3. CR $\Longleftrightarrow$ $\overset{*}{\leftrightarrow} = \downarrow$
4. CR $\Longrightarrow$ UN

Properties

1. T $\Longrightarrow$ N
2. T $\not\Longleftarrow$ N $\qquad\qquad \circlearrowleft a \longrightarrow b$
3. CR $\Longleftrightarrow$ $\overset{*}{\leftrightarrow} = \downarrow$
4. CR $\Longrightarrow$ UN
5. CR $\not\Longleftarrow$ UN

Properties

1. T $\Longrightarrow$ N
2. T $\not\Longleftarrow$ N $\quad \circlearrowright a \longrightarrow b$
3. CR $\Longleftrightarrow$ $\overset{*}{\leftrightarrow} = \downarrow$
4. CR $\Longrightarrow$ UN
5. CR $\not\Longleftarrow$ UN $\quad \circlearrowright a \longleftarrow b \longrightarrow c$

Properties

1. $\mathsf{T} \implies \mathsf{N}$
2. $\mathsf{T} \not\Longleftarrow \mathsf{N}$ $\qquad \circlearrowright a \longrightarrow b$
3. $\mathsf{CR} \iff \overset{*}{\leftrightarrow} = \downarrow$
4. $\mathsf{CR} \implies \mathsf{UN}$
5. $\mathsf{CR} \not\Longleftarrow \mathsf{UN}$ $\qquad \circlearrowright a \longleftarrow b \longrightarrow c$
6. $\mathsf{N} \wedge \mathsf{UN} \implies \mathsf{C}$

Properties

1. T $\Longrightarrow$ N
2. T $\not\Longleftarrow$ N $\qquad \circlearrowright a \longrightarrow b$
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7. C $\Longrightarrow$ LC

Properties

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8. C $\not\Longleftarrow$ LC

Properties

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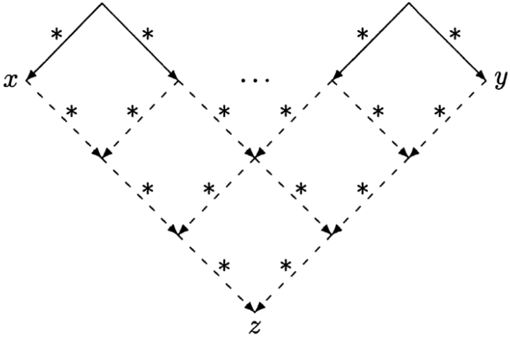
6. N $\wedge$ UN $\Longrightarrow$ C
7. C $\Longrightarrow$ LC
8. C $\not\Longleftarrow$ LC $\qquad a \longleftarrow b \rightleftarrows c \longrightarrow d$

Properties

- Recall what we were looking for.
- Ability to check equivalence by the search of a common reduct.
- This is exactly the Church-Rosser property.
- How does it relate to confluence and termination?

Church-Rosser and Confluence

- The Church-Rosser property and confluence coincide.
- CR $\Longrightarrow$ C is immediate.
- CR $\Longleftarrow$ C has a nice diagrammatic proof:



Central Notions: Semi-Confluence

Definition 1.5

A relation $\rightarrow$ is called semi-confluent (SC) iff

$$y_1 \leftarrow x \stackrel{*}{\rightarrow} y_2 \text{ implies } y_1 \downarrow y_2.$$

Central Notions: Semi-Confluence

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A relation $\rightarrow$ is called semi-confluent (SC) iff

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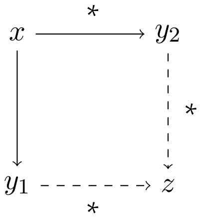
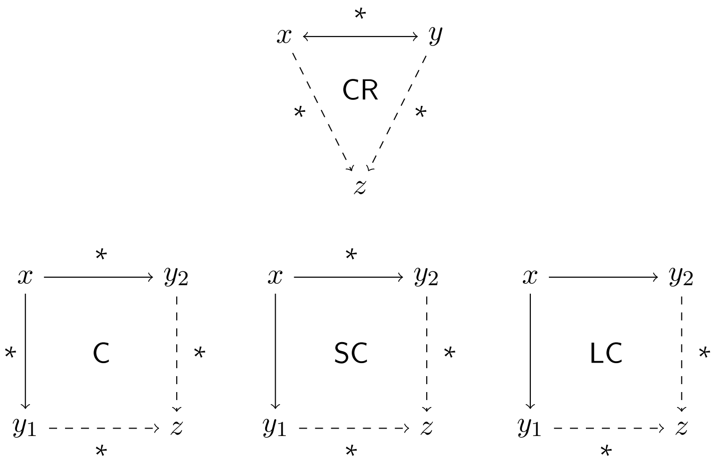
Graphically:



Solid arrows represent universal and dashed arrows existential quantification: $\forall x, y_1, y_2.\ y_1 \leftarrow x \xrightarrow{*} y_2 \Rightarrow \exists z.\ y_1 \xrightarrow{*} z \xleftarrow{*} y_2$.

CR, C, SC, LC



Church-Rosser, Confluence, and Semi-Confluence

Theorem 1.1

The following conditions are equivalent:

1. $\rightarrow$ *has the Chursh-Rosser property.*
2. $\rightarrow$ *is confluent.*
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Church-Rosser, Confluence, and Semi-Confluence

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The following conditions are equivalent:

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Proof.

$(1 \Rightarrow 2)$

□

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$(1 \Rightarrow 2)$

- Assume $\rightarrow$ is CR and $y_1 \xleftarrow{*} x \xrightarrow{*} y_2$. Show $y_1 \downarrow y_2$.

□

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- $y_1 \overset{*}{\leftarrow} x \overset{*}{\rightarrow} y_2$ implies $y_1 \overset{*}{\leftrightarrow} y_2$.

□

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- $y_1 \overset{*}{\leftarrow} x \overset{*}{\rightarrow} y_2$ implies $y_1 \overset{*}{\leftrightarrow} y_2$.
- CR implies $y_1 \downarrow y_2$.

□

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The following conditions are equivalent:

1. $\rightarrow$ *has the Chursh-Rosser property.*
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Proof.

$(2 \Rightarrow 3)$

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Proof.

$(2 \Rightarrow 3)$

- Semi-confluence is a special case of confluence.

$\square$

Church-Rosser, Confluence, and Semi-Confluence

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Church-Rosser, Confluence, and Semi-Confluence

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The following conditions are equivalent:

1. $\rightarrow$ *has the Chursh-Rosser property.*
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Proof.

$(3 \Rightarrow 1)$

□

Church-Rosser, Confluence, and Semi-Confluence

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The following conditions are equivalent:

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Proof.

$(3 \Rightarrow 1)$

- Assume $\rightarrow$ is SC and $x \overset{*}{\leftrightarrow} y$. Show $x \downarrow y$.

□

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The following conditions are equivalent:

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- Assume $\rightarrow$ is SC and $x \overset{*}{\leftrightarrow} y$. Show $x \downarrow y$.
- Induction on the length of the chain $x \overset{*}{\leftrightarrow} y$.

□

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- Assume $\to$ is SC and $x \overset{*}{\leftrightarrow} y$. Show $x \downarrow y$.
- Induction on the length of the chain $x \overset{*}{\leftrightarrow} y$.
- Base case: $x = y$. Trivial.

□

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- Assume $\rightarrow$ is SC and $x \overset{*}{\leftrightarrow} y$. Show $x \downarrow y$.
- Induction on the length of the chain $x \overset{*}{\leftrightarrow} y$.
- Base case: $x = y$. Trivial.
- Assume $x \overset{*}{\leftrightarrow} y' \leftrightarrow y$. Show $x \downarrow y$.

□

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- Base case: $x = y$. Trivial.
- Assume $x \overset{*}{\leftrightarrow} y' \leftrightarrow y$. Show $x \downarrow y$.
- By IH, $x \downarrow y'$, i.e. $x \overset{*}{\rightarrow} z \overset{*}{\leftarrow} y'$ for some z.

□

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$(3 \Rightarrow 1)$ (Cont.)

□

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Proof.

$(3 \Rightarrow 1)$ (Cont.)

- Show $x \downarrow y$ by case distinction on $y' \leftrightarrow y$.

□

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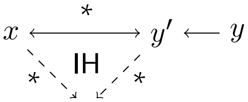
Proof.

$(3 \Rightarrow 1)$ (Cont.)

- Show $x \downarrow y$ by case distinction on $y' \leftrightarrow y$.
- $y' \leftarrow y$: $x \downarrow y$ follows directly from $x \downarrow y'$:



□

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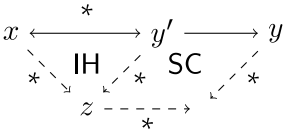
Proof.

$(3 \Rightarrow 1)$ (Cont.)

- Show $x \downarrow y$ by case distinction on $y' \leftrightarrow y$.
- $y' \rightarrow y$: Semi-confluence implies $z \downarrow y$ and, hence $x \downarrow y$:



□

Corollaries

- If $\rightarrow$ is confluent and $x \stackrel{*}{\leftrightarrow} y$ then
  1. $x \stackrel{*}{\rightarrow} y$ if y is in a normal form, and
  2. $x = y$ if both x and y are in a normal form.
- Hence, for confluent relations, convertibility is equivalent to joinability.
- Without termination, joinability can not be decided.

Corollaries

- If $\rightarrow$ is confluent, then every element has at most one normal form (C $\Longrightarrow$ UN)
- If $\rightarrow$ is normalizing and confluent, then every element has exactly one normal form.

Hence, for confluent and normalizing reductions the notation $x\downarrow$ is well-defined.

Goal-Directed Equivalence Test

Theorem 1.2

If $\rightarrow$ is confluent and normalizing, then

- *every element x has a unique normal form $x\downarrow$,*
- $x \overset{*}{\leftrightarrow} y$ *iff* $x\downarrow = y\downarrow$.

Normalization requires breadth-first search for normal forms.

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Theorem 1.3

If $\to$ is confluent and terminating, then

- *every element x has a unique normal form $x\downarrow$,*
- $x \overset{*}{\leftrightarrow} y$ *iff* $x\downarrow = y\downarrow$.

Termination permits depth-first search for normal forms.

Confluence and Termination

- How to show confluence and termination of an ARS?

Showing Termination

- Idea: Embedding the reduction into a well-founded order.

Showing Termination

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- Well-founded order $(B, >)$: No infinite descending chain $b_0 > b_1 > b_2 > \cdots$ in B.

Showing Termination

Examples of well-founded orders:

- $(\mathbb{N}, >)$: The set of natural numbers with the standard ordering.
- $(\mathbb{N} \setminus \{0\}, >)$: The set of positive integers where $a > b$ iff $b \mid a$ and $b \neq a$.
- $(\{a, b, c\}^*, >)$: The set of finite words over a fixed alphabet, where $w_1 > w_2$ iff w_2 is a proper substring of w_1.

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Examples of non-well-founded orders:

- $(\mathbb{Z}, >)$: The set of integers with the standard ordering.
- $(\mathbb{Q}_0^+, >)$: The set of non-negative rationals with the standard ordering.
- $(\{a, b, c\}^*, >)$: The set of finite words over a fixed alphabet, where $>$ is the lexicographic ordering, e.g. $a > ab > abb > \cdots$.

Showing Termination

Theorem 1.4

Let $(A, \to)$ *be an ARS. Then* $\to$ *is terminating iff there exists a well-founded order* $(B, >)$ *and a mapping* $\varphi : A \to B$ *such that*

$$a_1 \to a_2 \text{ implies } \varphi(a_1) > \varphi(a_2).$$

Showing Confluence (for a Terminating Relation)

Lemma 1.1 (Newman's Lemma)

If $\rightarrow$ is terminating and locally confluent, then it is confluent.

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Proof.

- Use well-founded induction. Let $(A, \to)$ be an ARS. Then WFI is the inference rule:

$$\frac{\forall x \in A.(\forall y \in A.(x \xrightarrow{+} y \Rightarrow P(y)) \Rightarrow P(x))}{\forall x \in A.P(x)} \text{ (WFI)}$$

  where P is some property of elements of A.
- Reads: To prove $P(x)$ for all $x \in A$, try to prove $P(x)$ under the assumption that $P(y)$ holds for all successors y of x.
- Holds when $\to$ is terminating.

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Proof. (Cont.)

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- Let P be

$$P(x) = \forall y, z.\ y \overset{*}{\leftarrow} x \overset{*}{\rightarrow} z \Rightarrow y \downarrow z.$$

  Obviously, $\rightarrow$ is confluent if $P(x)$ holds for all $x \in A$.

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- Fix x, y, z arbitrarily. Assume $y \overset{*}{\leftarrow} x \overset{*}{\rightarrow} z$. Prove $y \downarrow z$.
- Case 1: $x = y$ or $y = x$. Trivial.

Showing Confluence (for a Terminating Relation)

Lemma 1.1 (Newman's Lemma)

If $\rightarrow$ *is terminating and locally confluent, then it is confluent.*

Proof. (Cont.)

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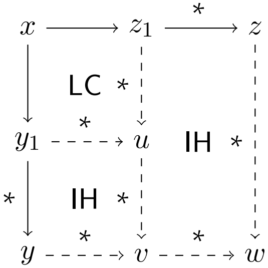

□

Showing Confluence (Termination Not Required)

Definition 1.6

A relation $\to$ is called strongly confluent (StC) iff

$$\forall x, y_1, y_2.\ \ y_1 \leftarrow x \to y_2 \Rightarrow \exists z.\ y_1 \overset{*}{\to} z \overset{=}{\leftarrow} y_2.$$

Remark: The definition is symmetric: $y_1 \leftarrow x \to y_2$ must imply both $y_1 \overset{*}{\to} z_1 \overset{=}{\leftarrow} y_2$ and $y_1 \overset{*}{\to} z_2 \overset{=}{\leftarrow} y_2$ for suitably chosen z_1 and z_2.

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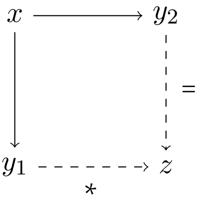
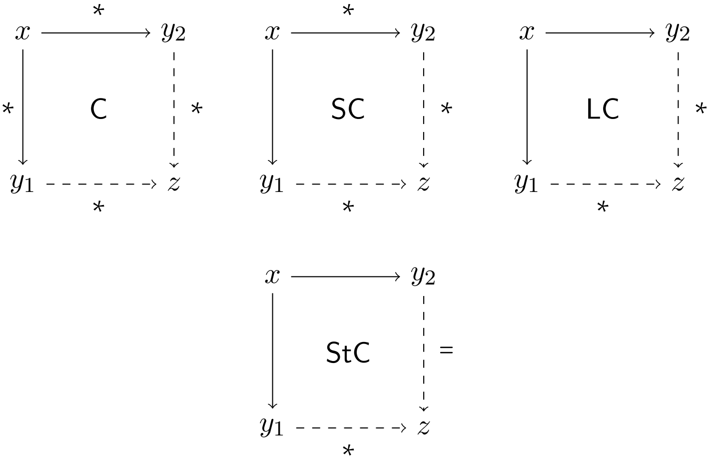
Graphically:



Solid arrows represent universal and dashed arrows existential quantification.

C, SC, LC, StC

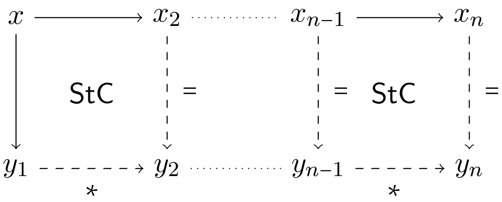


Showing Confluence (Termination Not Required)

Theorem 1.5

Any strongly confluent relation is semi-confluent (and, thus, confluent).

Proof.



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- If $\xrightarrow{*}_1 = \xrightarrow{*}_2$, then $\rightarrow_1$ is confluent iff $\rightarrow_2$ is confluent.
- Hence, if $\xrightarrow{*}_s = \xrightarrow{*}$, then $\mathsf{StC}(\rightarrow_s) \Rightarrow \mathsf{C}(\rightarrow_s) \Leftrightarrow \mathsf{C}(\rightarrow)$.

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- If $\xrightarrow{*}_1 = \xrightarrow{*}_2$, then $\to_1$ is confluent iff $\to_2$ is confluent.
- Hence, if $\xrightarrow{*}_s = \xrightarrow{*}$, then $\mathrm{StC}(\to_s) \Rightarrow \mathrm{C}(\to_s) \Leftrightarrow \mathrm{C}(\to)$.
- To simplify the search of $\to_s$, the condition can be weakened due to following easy lemma:

  If $\to_1 \subseteq \to_2 \subseteq \xrightarrow{*}_1$, then $\xrightarrow{*}_1 = \xrightarrow{*}_2$.

Showing Confluence (Termination Not Required)

Summarizing the ideas from the previous slide:

Theorem 1.6

If $\to \subseteq \to_s \subseteq \stackrel{*}{\to}$ *and* $\to_s$ *is strongly confluent, then* $\to$ *is confluent.*

Showing Confluence (Termination Not Required)

The theorem can be made stronger, considering the diamond property:

Definition 1.7

A relation $\rightarrow$ has the diamond property iff

$$\forall x, y_1, y_2.\;\; y_1 \leftarrow x \rightarrow y_2 \Rightarrow \exists z.\; y_1 \rightarrow z \leftarrow y_2.$$

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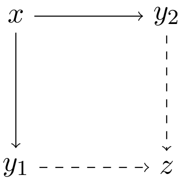

Showing Confluence (Termination Not Required)

The diamond property implies strong confluence, therefore:

Theorem 1.7

If $\rightarrow \subseteq \rightarrow_d \subseteq \xrightarrow{*}$ *and* $\rightarrow_d$ *has the diamond property, then* $\rightarrow$ *is confluent.*

Confluence by Commutation

- Confluence proofs can be localized by splitting a reduction up into several smaller reductions and showing their confluence separately.
- An additional property, commuting, should be satisfied.

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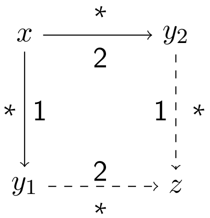
Definition 1.8

Two relations $\rightarrow_1$ and $\rightarrow_2$ commute iff

$$\forall x, y_1, y_2.\;\; y_1 \overset{*}{\leftarrow}_1 x \overset{*}{\rightarrow}_2 y_2 \Rightarrow \exists z.\; y_1 \overset{*}{\rightarrow}_2 z \overset{*}{\leftarrow}_1 y_2.$$



Confluence by Commutation

Lemma 1.2 (Commutative Union Lemma)

If $\to_1$ *and* $\to_2$ *are confluent and commute, then* $\to_1 \cup \to_2$ *is also confluent.*

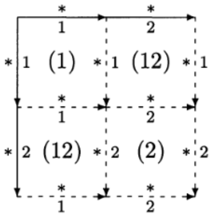
Confluence by Commutation

Lemma 1.2 (Commutative Union Lemma)

If $\rightarrow_1$ and $\rightarrow_2$ are confluent and commute, then $\rightarrow_1 \cup \rightarrow_2$ is also confluent.

Proof.

- $\overset{*}{\rightarrow}_1 \circ \overset{*}{\rightarrow}_2$ has the diamond property:



(1) Confluence of $\rightarrow_1$
(2) Confluence of $\rightarrow_2$
(12) Commutation of $\rightarrow_1$ and $\rightarrow_2$

Confluence by Commutation

Lemma 1.3 (Commutative Union Lemma)

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Confluence by Commutation

Lemma 1.3 (Commutative Union Lemma)

If $\rightarrow_1$ *and* $\rightarrow_2$ *are confluent and commute, then* $\rightarrow_1 \cup \rightarrow_2$ *is also confluent.*

Proof. (Cont.)

- The following inclusions hold:

$$\rightarrow_1 \cup \rightarrow_2 \subseteq \overset{*}{\rightarrow}_1 \circ \overset{*}{\rightarrow}_2 \subseteq (\rightarrow_1 \cup \rightarrow_2)^*.$$

- By Theorem 1.7, $\rightarrow_1 \cup \rightarrow_2$ is confluent.

□