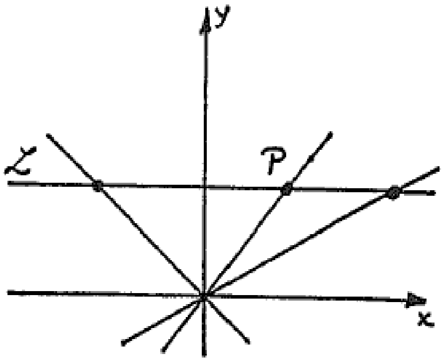
Chapter 5

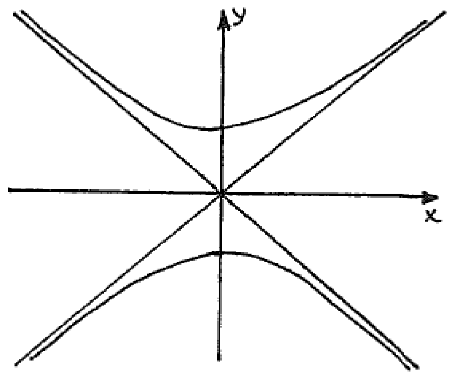
Projective algebraic sets and varieties

5.1 Projective space

In affine space we often have the problem that certain relations hold only up to some exceptions. So, for instance, two different lines meet in a point, except if they are parallel. A line intersects a hyperbola in two points, except if the line is one of the two asymptotes. These, and many other, exceptions in geometric statements can be eliminated by moving from affine to projective space.



model of $\mathbb{P}^1$ in $\mathbb{A}^2$



hyperbola

Figure 5.1

Consider Figure 5.1. Every point P on the line $\mathcal{L}$ is uniquely determined by the line through the origin and P. On the other hand, every such line through the origin determines a point P on $\mathcal{L}$, except for the line $y = 0$. We say that this line, which is parallel to $\mathcal{L}$, meets $\mathcal{L}$ "at infinity", i.e. it determines the "point at infinity", which we add to $\mathcal{L}$. The line $\mathcal{L}$ is just a copy of the affine line $\mathbb{A}^1$, and by adding this point at infinity we get what we call the projective line $\mathbb{P}^1$. Every point in $\mathbb{P}^1$ is uniquely determined by a line in $\mathbb{A}^2$ through the origin.

As another example, consider the hyperbola $y^2 = x^2 + 1$. Every line in $\mathbb{A}^2$ intersects the hyperbola in 2 points in $\mathbb{A}^2(\mathbb{C})$ (counted with multiplicity) except for the 2 lines $y = \pm x$. Proceeding in analogy to the treatment of 1–dimensional space above, we introduce a "point at infinity" in every direction. We identify a point $P = (a, b) \in \mathbb{A}^2$ with the line in $\mathbb{A}^3$ through $(a, b, 1)$ and $(0, 0, 0)$, written as $(a : b : 1)$ since only the proportionality of the coordinates is important. We get the coordinates of the point at infinity in the direction $\overline{OP}$ as

$$\lim_{n\to\infty}(na : nb : 1) \;=\; \lim_{n\to\infty}(a : b : \frac{1}{n}) \;=\; (a : b : 0).$$

The lines in $\mathbb{A}^3$ through $(a, b, 0)$ and $(0, 0, 0)$ correspond to the "points at infinity" in the projective plane $\mathbb{P}^2$.

Throughout this chapter we let K be an algebraically closed field.

Def. 5.1.1. Let $n \in \mathbb{N}_0$. For $(c_1, \ldots, c_{n+1}) \in \mathbb{A}^{n+1}(K) \setminus \{(0, \ldots, 0)\}$, by $(c_1 : \ldots : c_{n+1})$ we denote the line in $\mathbb{A}^{n+1}(K)$ through $(c_1, \ldots, c_{n+1})$ and $O = (0, \ldots, 0)$. So

$$(c_1 : \ldots : c_{n+1}) := \{\, (\lambda c_1, \ldots, \lambda c_{n+1}) \mid \lambda \in K\}.$$

The *n–dimensional projective space* $\mathbb{P}^n(K)$ over K is the set of all such lines through the origin in $\mathbb{A}^{n+1}$, i.e.

$$\mathbb{P}^n(K) := \{\, (c_1 : \ldots : c_{n+1}) \mid (c_1 : \ldots : c_{n+1}) \text{ line in } \mathbb{A}^{n+1}(K)\}.$$

The line $(c_1 : \ldots : c_{n+1})$ in $\mathbb{A}^{n+1}(K)$ is a *point* in $\mathbb{P}^n(K)$. The $(n+1)$–tuple $(c_1, \ldots, c_{n+1})$ is called (a *set of*) *homogeneous coordinates* of the point $(c_1 : \ldots : c_{n+1})$. •

Remark. $(c_1, \ldots, c_{n+1}), (d_1, \ldots, d_{n+1}) \in K^{n+1} \setminus \{O\}$ are homogeneous coordinates of the same point in projective space if and only if for some $\lambda \in K \setminus \{0\}$ we have

$$(c_1, \ldots, c_{n+1}) = \lambda(d_1, \ldots, c_{n+1}).$$

If we call such tuples equivalent, $c \sim d$, then

$$\mathbb{P}^n(K) = (K^{n+1} \setminus \{O\})_{/\sim}.$$

Remark. Let $K \subseteq \mathbb{C}$, $n \in \mathbb{N}_0$. Let π be the canonical surjective map

$$\begin{array}{rccc} \pi : & K^{n+1} \setminus \{O\} & \to & \mathbb{P}^n(K) \\ & (c_1, \ldots, c_{n+1}) & \mapsto & (c_1 : \ldots : c_{n+1})\ . \end{array}$$

The n–sphere $S^n(K) \subset K^{n+1} \setminus \{O\}$ is defined as

$$S^n(K) := \{\, c = (c_1, \ldots, c_{n+1}) \in K^{n+1} \mid \|c\|_2 = \sqrt{\sum_{i=1}^{n+1} c_i^2} = 1\,\}.$$

Consider the mapping

$$\begin{array}{rccc} \rho : & K^{n+1} \setminus \{O\} & \to & S^n(K) \\ & c = (c_1, \ldots, c_{n+1}) & \mapsto & \left(\frac{c_1}{\|c\|_2}, \ldots, \frac{c_{n_1}}{\|c\|_2}\right). \end{array}$$

We have the following commuting diagram, where π, ρ, π' are surjective:

$$\begin{tikzcd} K^{n+1} \setminus \{O\} \arrow[rr, "\pi"] \arrow[dr, "\rho"'] & & \mathbb{P}^n(K) \\ & S^n(K) \arrow[ur, "\pi' = \pi_{|S^n(K)}"'] & \end{tikzcd}$$

Example 5.1.1.

(a) The real projective line, $K = \mathbb{R}$, $n = 1$.

$$\begin{tikzcd} \mathbb{R}^2 \setminus \{O\} \arrow[rr, "\pi"] \arrow[dr, "\rho"'] & & \mathbb{P}^1(\mathbb{R}) \\ & S^1(\mathbb{R}) \arrow[ur, "\pi'"'] & \end{tikzcd}$$

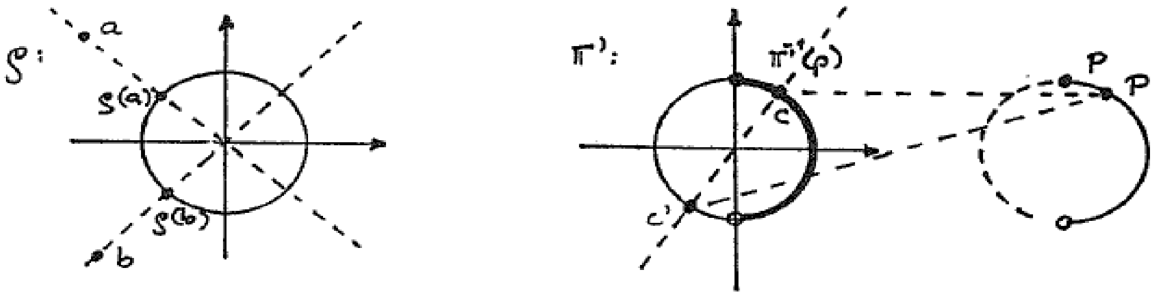


Figure 5.2

(b) The real projective plane, $K = \mathbb{R}$, $n = 2$.

$$\begin{tikzcd} \mathbb{R}^3 \setminus \{O\} \arrow[rr, "\pi"] \arrow[dr, "\rho"'] & & \mathbb{P}^2(\mathbb{R}) \\ & S^2(\mathbb{R}) \arrow[ur, "\pi'"'] & \end{tikzcd}$$

π' identifies the two points opposite to each other on the 2-sphere $S^2(\mathbb{R})$. So $\mathbb{P}^2(\mathbb{R})$ can be viewed as the upper half of S^2 plus half the equator. Lines in $\mathbb{P}^2(\mathbb{R})$ correspond to great circles on $S^2(\mathbb{R})$, so they always have a point of intersection. •

For a point $(c_1 : \ldots : c_{n+1}) \in \mathbb{P}^n$ the i–th coordinate c_i is not well–defined. But the property $c_i \neq 0$ is well–defined. And if $c_i \neq 0$, then the proportions c_j/c_i are well–defined.

Definition 5.1.2. (and remark) For $1 \leq i \leq n+1$ let $U_i := \{ (c_1 : \ldots : c_{n+1}) \in \mathbb{P}^n \mid c_i \neq 0 \}$. Then

$$\mathbb{P}^n = \bigcup_{i=1}^{n+1} U_i. \tag{$*$}$$

Every point $P \in U_i$ has uniquely determined coordinates of the form

$$P = (c_1 : \ldots : c_{i-1} : 1 : c_{i+1} : \ldots : c_{n+1}).$$

These coordinates $(c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_{n+1})$ are called the *non–homogeneous coordinates* of P w.r.t. U_i (or x_i, or i). The mapping

$$\begin{array}{cccc} \varphi_i : & \mathbb{A}^n & \to & U_i \\ & (a_1, \ldots, a_n) & \mapsto & (a_1 : \ldots : a_{i-1} : 1 : a_i : \ldots : a_n) \end{array}$$

is a bijection between $\mathbb{A}^n$ and U_i. Because of $(*)$ $\mathbb{P}^n$ can be covered by $n+1$ affine spaces of dimension n.
The set

$$H_\infty := \mathbb{P}^n \setminus U_{n+1} = \{ (c_1 : \ldots : c_n : 0) \}$$

is the *hyperplane at infinity* (w.r.t. x_{n+1} or $n+1$), the points on H_∞ are the *points at infinity*. Because of the correspondence $(c_1 : \ldots : c_n : 0) \leftrightarrow (c_1 : \ldots : c_n)$, the hyperplane at infinity H_∞ can be identified with $\mathbb{P}^{n-1}$. So we get

$$\mathbb{P}^n = U_{n+1} \cup H_\infty \sim \mathbb{A}^n \cup \mathbb{P}^{n-1},$$

i.e. $\mathbb{P}^n$ can be viewed as an n–dimensional affine space plus points at infinity corresponding to the directions in n–dimensional space. •

Example 5.1.2.

(a) Consider the line $y = ax + b$ in $\mathbb{A}^2(\mathbb{C})$. Identifying $\mathbb{A}^2$ with $U_3 \subset \mathbb{P}^2$, then the points on the line in affine space correspond to the points $(x : y : z) \in \mathbb{P}^2$ with $y = ax + bz$ and $z \neq 0$. Note that we have to make the defining polynomial homogeneous in order to make the solutions invariant under the equivalence $\sim$ of homogeneous coordinates.

$$\{\, (x : y : z) \in \mathbb{P}^2 \,|\, y = ax + bz \,\} \cap H_\infty = \{\, (1 : a : 0) \,\}.$$

So we see that the point at infinity $(1 : a : 0)$ is common to all lines of slope a. Two lines of the same slope intersect in a point at infinity.

(b) We reconsider the hyperbola $y^2 = x^2 + 1$. The corresponding set in $\mathbb{P}^2$ consists of the solutions of the homogeneous equation $y^2 = x^2 + z^2$ with $z \neq 0$. The set $\{\, (x : y : z) \in \mathbb{P}^2 \,|\, y^2 = x^2 + z^2 \,\}$ intersects H_∞ in the two points $(1 : 1 : 0)$ and $(1 : -1 : 0)$. These are the points in which the hyperbola intersects the asymptotic lines $y = x$ and $y = -x$, respectively.

(c) Consider circles $\mathcal{C}_a : x^2 + y^2 - a = 0$ centered at the affine origin, and with different radii. In affine space, even over the algebraically closed field $\mathbb{C}$, these circles (for different radii) have no intersection. But if we view them in the projective plane, defined by the homogeneous polynomials $x^2 + y^2 - az^2 = 0$, we find that all of them share the points at infinity $(1 : \pm i : 0)$. In fact (as we will be able to determine later) all these circles are tangent to the line at infinity at these so-called *circular points*. •

5.2 Homogeneous ideals and projective algebraic sets

Def. 5.2.1. A *form* or *homogeneous polynomial* in the polynomial ring $R[x_1, \ldots, x_n]$ over the ring R is a polynomial, in which every term has the same degree.

Let $f = f_0 + f_1 + \ldots f_d \in R[x_1, \ldots, x_n]$ be such that the f_i, $0 \leq i \leq d$, are forms of degree i, respectively. Then

$$f^* = f_0 x_{n+1}^d + f_1 x_{n+1}^{d-1} + \ldots + f_d = x_{n+1}^d \cdot f(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}})$$

is a form of degree d, and it is called the *homogenization* of f w.r.t. x_{n+1}.

Conversely, if $F \in R[x_1, \ldots, x_{n+1}]$ is a form, then

$$F_* = F(x_1, \ldots, x_n, 1) \in R[x_1, \ldots, x_n]$$

is the *dehomogenization* of F w.r.t. x_{n+1}. •

In fact, F_* also makes sense for non-homogeneous polynomials. The proof of the following theorem is rather obvious and is left to the reader.

Theorem 5.2.1. *Let $f, g \in R[x_1, \ldots, x_n]$, F, G forms in $R[x_1, \ldots, x_n, x_{n+1}]$.*

(a) $(F \cdot G)_* = F_* \cdot G_*$ *and* $(f \cdot g)^* = f^* \cdot g^*$.

(b) *If r is the highest power such that $x_{n+1}^r \mid F$, then $x_{n+1}^r (F_*)^* = F$.*
$(f^*)_* = f$.

(c) $(F + G)_* = F_* + G_*$.
$x_{n+1}^t (f + g)^* = x_{n+1}^r f^* + x_{n+1}^s g^*$,
where $r = \deg(g), s = \deg(f), t = r + s - \deg(f + g)$.

The second equation in part (c) of the previous theorem is not minimal. In fact, it can be divided by $x_{n+1}^{\min(r,s)}$.

Def. 5.2.2. $P \in \mathbb{P}^n$ is a *root* of the polynomial $f(x_1, \ldots, x_{n+1}) \in K[x_1, \ldots, x_{n+1}]$ iff $f(c_1, \ldots, c_{n+1}) = 0$ for every choice of homogeneous coordinates $(c_1 : \ldots : c_{n+1})$ of P. In this case we write $f(P) = 0$.
For $S \subseteq K[x_1, \ldots, x_{n+1}]$ we define

$$V_p(S) := \{ P \in \mathbb{P}^n \mid f(P) = 0 \text{ for all } f \in S \}.$$

$V_p(S)$ is a *projective algebraic set.* •

Usually we will just write $V(S)$ instead of $V_p(S)$, if it is clear from the context that we mean a projective algebraic set. In particular, this holds for this chapter.

Lemma 5.2.2. *Let K be an infinite field, $f \in K[x_1, \dots, x_{n+1}]$, $f_0, \dots, f_d$ forms with $\deg(f_i) = i$ such that $f = \sum_{i=0}^{d} f_i$. $P \in \mathbb{P}^n(K)$ is a root of f if and only if P is a root of f_i for all $0 \le i \le d$.*

Proof: If P is a root of every f_i, then obviously it is also a root of f.

Conversely, let $(c_1 : \dots : c_{n+1})$ be a fixed tuple of homogeneous coordinates of P. We consider the polynomial

$$g(\lambda) = f(\lambda c_1, \dots, \lambda c_{n+1}) = \sum_{i=0}^{d} \lambda^i \cdot f_i(c_1, \dots, c_{n+1}).$$

For P to be a root of f, the polynomial g must vanish on all $\lambda \in K \setminus \{0\}$. Since K is infinite, this is only possible if $g = 0$, i.e. $f_i(c_1, \dots, c_{n+1}) = 0$ for all $0 \le i \le d$. •

If $I = \langle f^{(1)}, \dots, f^{(r)} \rangle$, where $f^{(i)} = \sum f_j^{(i)}$, $f_j^{(i)}$ a form of degree j, then $V_p(I) = V(I) = V(\{f_j^{(i)}\})$. So every projective algebraic set is the set of roots of a finite set of forms.

Def. 5.2.3. For a set $X \subseteq \mathbb{P}^n$ let

$$I(X) := \{\, f \in K[x_1, \dots, x_{n+1}] \mid f(P) = 0 \text{ for every } P \in X \,\}.$$

$I(X)$ is the *ideal of* X. •

If $f \in I(X)$ then, by Lemma 5.2.2, all homogeneous components of f are contained in $I(X)$.

Def. 5.2.4. The ideal $I \subseteq K[x_1, \dots, x_{n+1}]$ is *homogeneous* iff, for every

$$f = \sum_{i=0}^{d} f_i \in I, \quad f_i \text{ form of degree } i,$$

also $f_i \in I$ for $0 \le i \le d$. •

So for every $X \subseteq \mathbb{P}^n$, the ideal $I(X)$ is homogeneous.

Lemma 5.2.3. (a) *A homogeneous ideal $I \subseteq K[x_1, \dots, x_{n+1}]$ is prime if and only if*

$$f \cdot g \in I \Longrightarrow f \in I \ \vee \ g \in I,$$

for arbitrary forms $f, g \in K[x_1, \dots, x_{n+1}]$.
(b) *If I is homogeneous, then also $\sqrt{I}$ is homogeneous.* •

Proof: (a) Clearly any prime ideal must contain one of the forms, if their product is in I; so, a forteriori, also a homogeneous prime ideal.
On the other hand, we have to show, that for a homogeneous ideal the validity of the

product rule for forms implies the validity of the product rule for arbitrary polynomials. To see this, assume that there exist polynomials f, g s.t.

$$f \cdot g \in I, \quad \text{but} \quad f, g \notin I. \tag{$*$}$$

Let f, g be such that $\deg(f \cdot g)$ is least with this property. Write

$$\begin{aligned} f &= f_k + \cdots + f_0, \\ g &= g_l + \cdots + g_0, \end{aligned}$$

where f_i, g_i are forms of degree i, and $f_k \neq 0 \neq g_l$.
Since I contains $f \cdot g$, is must also contain its highest degree form $f_k \cdot g_l$, and therefore either f_k or g_l. W.l.o.g. assume $f_k \in I$. Then also $(f_{k-1} + \cdots f_0) \cdot g = f \cdot g - f_k \cdot g \in I$, and it is of lower degree than $f \cdot g$. So either $(f_{k-1} + \cdots f_0) \in I$, and therefore $f \in I$, or $g \in I$. In either case we arrive at a contradiction to $(*)$.
(b) Let f be written as in (a). It suffices to show that $f \in \sqrt{I}$ implies $f_k \in \sqrt{I}$. From $f \in \sqrt{I}$ we get $f^m = f_k^m +$lower degree forms $\in I$ for some m, so $f_k^m \in I$, and therefore $f_k \in \sqrt{I}$. •

Theorem 5.2.4. *The ideal $I \subseteq K[x_1, \ldots, x_{n+1}]$ is homogeneous if and only if it is generated by a (finite) set of forms.*

Proof: "$\Longrightarrow$": If $I = \langle f^{(1)}, \ldots, f^{(r)} \rangle$ is homogeneous, $f^{(i)} = \sum f_j^{(i)}$, where $f_j^{(i)}$ are forms of the respective degrees, then $I = \langle \{f_j^{(i)}\} \rangle$.
"$\Longleftarrow$": Let $I = \langle S \rangle$, $S = \{f^{(i)}\}$ a set of forms with $\deg(f^{(i)}) = d_i$. Consider an arbitrary

$$g = g_m + g_{m-1} + \ldots + g_r \in I, \quad \deg(g_i) = i.$$

It suffices to show $g_m \in I$, for then also $g - g_m \in I$, and the statement follows by induction. For some polynomials $a^{(i)}$ we have

$$g = \sum a^{(i)} f^{(i)}.$$

Comparing terms of the same degree on both sides of this equations, we get

$$g_m = \sum_{d_i \leq m} a_{m-d_i}^{(i)} \cdot f^{(i)}.$$

So $g_m \in I$. •

As for affine sets, we call a projective algebraic set $V \subseteq \mathbb{P}^n$ *irreducible* iff it cannot be written as the union of two proper algebraic subsets. An irreducible projective algebraic set is a *projective variety.* Analogously to the affine case one proofs that every projective algebraic set can be decomposed uniquely into a union of finitely many projective varieties. These are called the *irreducible components* of the projective algebraic set.

Furthermore, in analogy to the affine case, one shows (using Lemma 5.2.3.(a)) that the projective algebraic set V is irreducible if and only if $I(V)$ is prime.

For the (projective) mappings

$$\begin{array}{ccc} \text{homogeneous ideals} & \overset{V}{\longrightarrow} & \text{algebraic sets} \\ K[x_1,\ldots,x_{n+1}] & \underset{I}{\longleftarrow} & \text{in } \mathbb{P}^n(K) \end{array}$$

we again have the facts (1) — (6) of Section 3.1., and also the properties analogous to the ones in Lemma 3.1.3.

If confusion might arise, we write V_p, I_p for the projective mappings, and V_a, I_a for the affine ones.

Def. 5.2.5. For a projective algebraic set $V \subseteq \mathbb{P}^n$ we call

$$C(V) := \{\, (c_1,\ldots,c_{n+1}) \in \mathbb{A}^{n+1} \,|\, (c_1 : \ldots : c_{n+1}) \in V \,\} \;\cup\; \{(0,\ldots,0)\}$$

the *cone over* V. •

From this definition we see that for a homogeneous ideal I we have

$$C(V_p(I)) = V_a(I) \cup \{\mathcal{O}\}.$$

Cones allow the transfer of properties of affine algebraic sets into properties of projective algebraic sets, and vice versa.

Lemma 5.2.5.

(a) *If K is infinite, $\emptyset \neq V \subseteq \mathbb{P}^n(K)$, then $I_a(C(V)) = I_p(V)$.*

(b) *If I is a homogeneous ideal in $K[x_1,\ldots,x_{n+1}]$ with $V_p(I) \neq \emptyset$, then $C(V_p(I)) = V_a(I)$.* •

Proof: (a) $f(x_1,\ldots,x_{n+1}) \in I_a(C(V)) \iff$
$f(\lambda a_1,\ldots,\lambda a_{n+1}) = 0$ for all $P = (a_1 : \ldots : a_{n+1}) \in V$ (and $f_0 = 0$) $\iff$
f is homogeneous and vanishes on all $P \in V$.

(b) Because of the observation after Def. 5.2.5 we only need to check whether $\mathcal{O} \in V_a(I)$.
Let $Q \in V_p(I)$. Then any $f \in I$ vanishes on Q; so the constant component of f is 0, and therefore $\mathcal{O} \in V_a(I)$. •

The exceptions in Lemma 5.2.5 are really necessary. In part (a), if $V = \emptyset$, then $I_p(V) = \langle 1\rangle$, but $I_a(C(V)) = I_a(\{\mathcal{O}\}) = \langle x_1,\ldots,x_{n+1}\rangle$.
In part (b), if we consider $I = \langle 1\rangle$, then $V_p(I) = \emptyset$; $C(V_p(I)) = \{\mathcal{O}\}$, but $V_a(I) = \emptyset$.

Theorem 5.2.6. (Projective Nullstellensatz) *Let K be an algebraically closed field, I a homogeneous ideal in $K[x_1,\ldots,x_{n+1}]$.*

(a) $V_p(I) = \emptyset \quad \Longleftrightarrow \quad I$ *contains all forms of degree* $\geq N$*, for some* $N \in \mathbb{N}$.

(b) *If* $V_p(I) \neq \emptyset$*, then* $I_p(V_p(I)) = \sqrt{I}$.

Proof: (a)

$$\begin{array}{l} V_p(I) = \emptyset \quad \Longleftrightarrow \\ V_a(I) \subseteq \{(0,\dots,0)\} \quad \Longleftrightarrow_{\uparrow \text{ by the affine Nullstellensatz}} \\ \sqrt{I} = I_a(V_a(I)) \supseteq \langle x_1,\dots,x_{n+1}\rangle \quad \Longleftrightarrow \\ \langle x_1,\dots,x_{n+1}\rangle^N \subseteq I \text{ for some } N \in \mathbb{N}. \end{array}$$

(b)

$$\begin{array}{l} I_p(V_p(I)) =_{\uparrow \text{ Lemma 5.2.5.(a)}} \\ I_a(C(V_p(I))) =_{\uparrow \text{ Lemma 5.2.5.(b)}} \\ I_a(V_a(I)) =_{\uparrow \text{ aff. Nullst.satz}} \\ \sqrt{I}. \end{array}$$

•

The usual corollaries to the Nullstellensatz also hold for the projective case. However, we always have to exclude the ideal $\langle x_1,\dots,x_{n+1}\rangle$.

Def. 5.2.6. An *affine change of coordinates* is an invertible linear mapping

$$\begin{array}{cccc} T: & \mathbb{A}^n & \longrightarrow & \mathbb{A}^n \\ & x & \mapsto & A\cdot x + b, \end{array}$$

i.e. A is an invertible matrix.

An affine change of coordinates of the form

$$\begin{array}{cccc} T: & \mathbb{A}^{n+1} & \longrightarrow & \mathbb{A}^{n+1} \\ & x & \mapsto & A\cdot x \end{array}$$

where A is an invertible $(n+1)\times(n+1)$–matrix, transforms lines through the origin into lines through the origin. So T determines a mapping from $\mathbb{P}^n$ to $\mathbb{P}^n$. Such a mapping is called a *projective change of coordinates.* •

The proofs of the following statements are left as exercises.

Theorem 5.2.7. *Let* $V \subseteq \mathbb{P}^n$ *be a set of points,* $T = (T_1,\dots,T_{n+1})$ *a projective change of coordinates in* $\mathbb{P}^n$.

(a) V *is algebraic if and only if* $V^T := T^{-1}(V)$ *is algebraic.*

(b) *If* $V = V(f_1,\dots,f_r)$*, then* $V^T = V(f_1^T,\dots,f_r^T)$*, where* $f_i^T = f_i(T_1,\dots,T_{n+1})$.

(c) V *is a variety if and only if* V^T *is a variety.* •

Lemma 5.2.8. *Let P_1, P_2, P_3 and Q_1, Q_2, Q_3 be non–collinear points in $\mathbb{P}^2$, respectively. Then there is a projective change of coordinates $T : \mathbb{P}^2 \to \mathbb{P}^2$ such that $T(P_i) = Q_i$ for $1 \le i \le 3$.* •

By way of the mapping

$$\varphi_{n+1} : \mathbb{A}^n \to U_{n+1} \subset \mathbb{P}^n$$

(compare Def. 5.1.2) the affine space $\mathbb{A}^n$ can be viewed as a subset of the projective space $\mathbb{P}^n$. In the sequel we want to investigate the relations between algebraic sets in $\mathbb{A}^n$ and $\mathbb{P}^n$.

As we can homogenize and dehomogenize polynomials, we can do the same for polynomial ideals.

Def. 5.2.7. For an ideal $I \subseteq K[x_1, \ldots, x_n]$ we call $I^* := \langle \{f^* | f \in I\} \rangle \subseteq K[x_1, \ldots, x_{n+1}]$ the *homogenization* of I.
For a homogeneous ideal $I \subseteq K[x_1, \ldots, x_{n+1}]$ we call $I_* := \{f_* | f \in I\} \subseteq K[x_1, \ldots, x_n]$ the *dehomogenization* of I (w.r.t. x_{n+1}). •

Lemma 5.2.9.

(i) *For an ideal I in $K[x_1, \ldots, x_n]$, the homogenization I^* is a homogeneous ideal in $K[x_1, \ldots, x_{n+1}]$.*

(ii) *For a homogeneous ideal $I \subseteq K[x_1, \ldots, x_{n+1}]$, the dehomogenization $I_* := \{f_* | f \in I\} \subseteq K[x_1, \ldots, x_n]$ is an ideal in $K[x_1, \ldots, x_n]$.*

Def. 5.2.8. For an algebraic set V in $\mathbb{A}^n$ we define the *projective closure* of V as $V^* := V_p(I(V)^*)$ in $\mathbb{P}^n$.
Conversely, for an algebraic set V in $\mathbb{P}^n$ we define the *restriction to affine space* of V as $V_* := V_a(I(V)_*)$ in $\mathbb{A}^n$. •

Example 5.2.1. Consider the hyperbola, the curve $\mathcal{C}$ defined by $y^2 - x^2 - 1 = 0$ in $\mathbb{A}^2$.
The projective closure of $\mathcal{C}$ is $\mathcal{C}^* = V_p(I(\mathcal{C})^*) = \mathcal{C} \cup \{(1 : \pm 1 : 0)\}$.
Conversely, $(\mathcal{C}^*)_* = V_a(I(\mathcal{C}^*)_*) = \mathcal{C}$. •

Theorem 5.2.10. *Let V, W be algebraic sets.*

(1) *If $V \subseteq \mathbb{A}^n$ then $\varphi_{n+1}(V) = V^* \cap U_{n+1}$ and $(V^*)_* = V$.*

(2) *If $V \subseteq W \subseteq \mathbb{A}^n$ then $V^* \subseteq W^* \subseteq \mathbb{P}^n$.*
If $V \subseteq W \subseteq \mathbb{P}^n$ then $V_* \subseteq W_* \subseteq \mathbb{A}^n$.

(3) *If V is irreducible in $\mathbb{A}^n$ then V^* is irreducible in $\mathbb{P}^n$.*

(4) *For* $V \subseteq \mathbb{A}^n$*, the projective closure* V^* *is the smallest algebraic set in* $\mathbb{P}^n$ *containing* V $(= \varphi_{n+1}(V))$.

(5) *If* $V \subseteq \mathbb{A}^n$ *and* $V = \bigcup_{i=1}^r V_i$ *is the irreducible decomposition of* V*, then* $V^* = \bigcup_{i=1}^r V_i^*$ *is the irreducible decomposition of* V^* *in* $\mathbb{P}^n$.

(6) *If* $V \subset \mathbb{A}^n, V \neq \mathbb{A}^n$*, then there is no component* V_i^* *of* V^* *such that* $V_i^* \subseteq H_\infty = \mathbb{P}^n \setminus U_{n+1}$ *or* $H_\infty \subseteq V_i^*$.

(7) *If* $V \subseteq \mathbb{P}^n$ *and there is no component* V_i *of* V *such that* $V_i \subseteq H_\infty$ *or* $H_\infty \subseteq V_i$*, then* $V_* \subset \mathbb{A}^n, V_* \neq \mathbb{A}^n$*, and* $(V_*)^* = V$.

Proof: (1) follows from Theorem 5.2.1.
(2) is obvious.
(3): Let $I = I(V)$. For a form $F \in K[x_1, \ldots, x_{n+1}]$ we have $F \in I^* \iff F_* \in I$. By Theorem 5.2.1 a product of forms $F \cdot G$ is in I^* if and only if $F_* \cdot G_*$ is in I, and by the primality of I this leads to one of the factors F^*, G^* being in I^*. So if I is prime, then also I^* is prime.
(4): Let W be algebraic in $\mathbb{P}^n$ such that $\varphi_{n+1}(V) \subseteq W$. If $F \in I(W)$ then $F_* \in I(V)$, so $F = x_{n+1}^r (F_*)^* \in I(V)^*$. Thus, $I(W) \subseteq I(V)^*$, and therefore $W \supseteq V^*$.
(5): follows from (2), (3), and (4).
(6): w.l.o.g. we can assume that V is irreducible. $V^* \not\subseteq H_\infty$ by (1). If $H_\infty \subseteq V^*$, then we would have (by (4)) $I(V)^* \subseteq I(V^*) \subseteq I(H_\infty) = \langle x_{n+1} \rangle$. But for $f \in I(V) \setminus \{0\}$ we have $f^* \in I(V)^*$ and $f^* \notin \langle x_{n+1} \rangle$. Thus, $H_\infty \not\subseteq V^*$.
(7): Again, w.l.o.g. we can assume that V is irreducible. $\varphi_{n+1}(V_*) \subseteq V$. $(V_*)^*$ is the smallest variety containing V_* (by (4)). So $(V_*)^* \subseteq V$.
Conversely, we have to show $V \subseteq (V_*)^*$, or $I((V_*)^*) \subseteq I(V)$. Observe that $I(V_*)^*$ is radical, so $I((V_*)^*) = I(V(I(V_*)^*)) = I(V_*)^*$. Let $f \in I(V_*)$. By the Nullstellensatz $f^N \in \sqrt{I(V_*)} = I(V(I(V(I(V)_*)))) = I(V)_*$, for some N. So, by Theorem 5.2.1, $x_{n+1}^t \cdot (f^N)^* \in I(V)$ for some t. But $I(V)$ is prime and $x_{n+1} \notin I(V)$ (since $V \not\subseteq H_\infty$), therefore $f^* \in I(V)$. •

Furthermore, one can show:

– If $H_\infty \subseteq V \subseteq \mathbb{P}^n$, V a variety, then $V = \mathbb{P}^n$ or $V = H_\infty$,

– if $V = \mathbb{P}^n$, then $V_* = \mathbb{A}^n$,

– if $V = H_\infty$, then $V_* = \emptyset$.

So there is a 1-1 correspondence between affine varieties and projective varieties not contained in H_∞.

For plane curves these observations specialize in the following way: Let $\mathcal{C} = V(f) \subset \mathbb{A}^2$ be an affine plane curve defined by $f(x, y) = 0$. The corresponding projective curve is $\mathcal{C}^* = V(f)^* = V(f^*)$, where

$$f^* = f_d(x,y) + f_{d-1}(x,y) \cdot z + \ldots + f_0(x,y) \cdot z^d,$$

the homogenization of f. Every point $(a,b) \in \mathcal{C}$ corresponds to a point $(a:b:1) \in \mathcal{C}^*$, and every additional point on $\mathcal{C}^*$ is a point at infinity, i.e. a solution of $f_d(x,y) = 0, z = 0$. So the curve $\mathcal{C}^*$ has only finitely many points at infinity.