Chapter 2

Rational general solutions of first-order algebraic ODEs

We recall the notion of a general solution of a first-order algebraic ordinary differential equation (ODE) from the point of view of differential algebra, i.e., it is defined as a generic zero of a prime differential ideal in a differential ring. We refer to the appendix section (B) on differential algebra for most of preliminary notions that need for this chapter. The main development of this chapter is the algebraic geometric method for determining a rational general solution of a first-order algebraic ODE. We observe that the solution surface of an algebraic ODE of order 1 having a rational general solution must be a unirational surface. By Castelnuovo's theorem, every unirational surface over an algebraically closed field of characteristic 0 (e.g. the field of complex numbers) is a rational surface. Therefore, we only consider the class of all first-order parametrizable algebraic ODEs, i.e., the differential equation $F(x, y, y') = 0$ such that $F(x, y, z) = 0$ defines a rational surface. This class naturally extends the class of first-order autonomous ODEs in Feng and Gao (2004, 2006). In this class, we derive an associated system from a proper rational parametrization of the solution surface of the given differential equation. Then we prove that there is a one-to-one correspondence between a rational general solution of the given first-order parametrizable algebraic ODE and that of its associated system. In the last section, we give a criterion, based on Ritt's reduction of Feng-Gao's differential polynomials, for the existence of a rational general solution of the associated system. As an application, we use the criterion for determining the linear systems with rational general solutions.

Throughout this chapter, we consider $\mathbb{K}$ to be an algebraically closed field of characteristic zero, i.e., $\mathbb{K}$ contains the field of rational numbers $\mathbb{Q}$. The content of this chapter is essentially based on Ngô and Winkler (2010).

2.1 Definition of (rational) general solutions

Given an algebraic ODE $F(x, y', y'', \cdots, y^{(n)}) = 0$ of order n, where F is a polynomial over $\mathbb{K}$. Classically, a solution of $F(x, y', y'', \cdots, y^{(n)}) = 0$ is a function depending on x, $y = f(x)$, such that $F(x, f(x), f'(x), \cdots, f^{(n)}(x)) = 0$. From the point of view of algebra, the polynomial F can be formally seen as a differential polynomial in the differential ring $(\mathbb{K}(x)\{y\}, \delta)$, where y is a differential indeterminate and δ is the unique derivation extended from the usual derivation $\frac{d}{dx}$ of the differential field $\mathbb{K}(x)$ (see B). Let us write the above differential equation in the form $F(y) = 0$ to simplify the notation when we do not want to stress on the order of the equation.

Let $(\mathcal{K}, \delta)$ be a differential field extension of $(\mathbb{K}(x), \frac{d}{dx})$. A solution of $F(y) = 0$ in $\mathcal{K}$ is an element $\eta \in \mathcal{K}$ such that $F(x, \eta, \delta\eta, \cdots, \delta^n\eta) = 0$. Observe that, if η is a solution of $F(y) = 0$, then η is also a solution of all $\delta^m(F)(y) = 0$ for any natural number $m \geq 1$. In fact, η is also a solution of the differential ideal generated by F, denoted by $[F]$. Furthermore, according to the theorem of zeros, Ritt (1950), II, §7 (also known as the differential Nullstellensatz), the collection of all differential polynomials in $\mathbb{K}(x)\{y\}$ vanishing on the solutions of F is the *radical differential ideal* generated by F, denoted by $\{F\}$ the set

$$\{F\} = \{A \in \mathbb{K}(x)\{y\} \mid \exists m \in \mathbb{N},\ A^m \in [F]\}.$$

It is known from Ritt (1950), II, §14, that we can decompose $\{F\}$ as

$$\{F\} = (\{F\} : S) \cap \{F, S\}, \tag{2.1}$$

where S is the separant of F and $\{F\} : S = \{A \in \mathbb{K}(x)\{y\} \mid SA \in \{F\}\}$. Note that $\{F\} : S$ is a radical differential ideal and $\{F\} : S = \{F\} : S^\infty$, defined by

$$\{F\} : S^\infty = \{A \in \mathbb{K}(x)\{y\} \mid \exists m \in \mathbb{N},\ \ S^m A \in \{F\}\}.$$

The ideal $\{F\} : S^\infty$ is called the saturation ideal of $\{F\}$ by S. Moreover, if F is an irreducible polynomial in the polynomial ring $\mathbb{K}[x, y_1, y_2, \ldots, y_n]$, which we assume from now on, then $\{F\} : S$ is a prime differential ideal (by Ritt (1950), II, §12). One can further decompose $\{F, S\}$ as the intersection of finite number of prime differential ideals because $\mathbb{K}(x)\{y\}$ is a radical Noetherian ring, i.e., the ring in which every radical differential ideal is finitely generated. In the end, if we exclude all redundant prime differential ideals, then we obtain a unique minimal decomposition of $\{F\}$ into an intersection of irredundant prime differential ideals, which are called *essential components* of $\{F\}$. Furthermore, $\{F\} : S$ is the unique essential component of $\{F\}$ that does not contain the separant S because if $\{F\} : S$ would contain S, then $S^2 \in \{F\}$. Hence, $S \in \{F\}$, a contradiction to the fact that $\deg_{y^{(n)}} S < \deg_{y^{(n)}} F$.

Definition 2.1.1. A generic zero of the prime differential ideal $\{F\} : S$ is called a *general solution* of $F(y) = 0$. By a *generic zero* η of $\{F\} : S$ we mean for all $G \in \mathbb{K}(x)\{y\}$, $G(\eta) = 0 \Longleftrightarrow G \in \{F\} : S$.

Definition 2.1.2. A zero of $\{F, S\}$ is called a *singular solution* of $F(y) = 0$.

Of course, when we decompose $\{F, S\}$ into prime differential ideals, there might be some components that contain $\{F\} : S$. These components will be corresponding to the *particular solutions* of $F(y) = 0$ in the classical sense. We will demonstrate this in an example later.

In the quotient ring $\mathbb{K}(x)\{y\}/(\{F\} : S)$, which is an integral domain*, the class of y is a generic zero of the prime differential ideal $\{F\} : S$. However, this is still an implicit description of a general solution of $F(y) = 0$. Another way of describing a general solution of $F(y) = 0$ is computing a basis of the differential ideal $\{F\} : S$. In her paper Hubert (1996), Hubert presents an algorithmic method to determine a Gröbner basis of the differential ideal $\{F\} : S$ in the case $\mathrm{ord}(F) = 1$. Our question is:

> **Q 1.** *How to construct a generic zero of* $\{F\} : S$*, i.e., a general solution of* $F(y) = 0$ *explicitly?*
>
> Our goal, in this chapter and in the next chapter(s), is to develop a method to construct explicitly a rational general solution of $F(y) = 0$ in the case of *first-order parametrizable algebraic ODEs*, $F(x, y, y') = 0$, where $F(x, y, z) = 0$ defines a rational surface.

Definition 2.1.3. A *rational general solution* of $F(y) = 0$ is defined as a general solution of $F(y) = 0$ of the form

$$y = \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0}, \tag{2.2}$$

where a_i, b_j are constants in a differential field extension of $\mathbb{K}(x)$.

From the definition of a general solution of $F(y) = 0$, it is important to know when a differential polynomial belongs to $\{F\} : S$. This ideal membership problem is solved by using Ritt's reduction. Precisely, consider the differential ring $\mathbb{K}(x)\{y\}$ with the orderly ranking (see B.2). Then for any $G \in \mathbb{K}(x)\{y\}$, we have

$$S^{s_F} I^{i_F} G = \sum_{i \geq 0} Q_i \delta^i(F) + R, \tag{2.3}$$

*In Ritt (1950), II, §6, one can define a natural derivation on the quotient field of this integral domain so that it becomes a differential field.

where $\delta^i(F)$ is the i-th derivative of F and the order of $\delta^i(F)$ is less than or equal the order of G; S is the *separant* of F, I is the *initial* of F, $s_F, i_F \in \mathbb{N}$, $Q_i \in \mathbb{K}(x)\{y\}$ and $R \in \mathbb{K}(x)\{y\}$ is *reduced* with respect to F, i.e., if n is the order of F, then the order of R is at most n and $\deg_{y^{(n)}}(R) < \deg_{y^{(n)}}(F)$. Moreover, if m is the order of G, then $0 \le i \le m - n$. By convention, the sum is empty if $m < n$; in this case, $R = G$.

Definition 2.1.4. The differential polynomial R in (2.3) is called the *differential pseudo remainder* of G with respect to F, denoted by $\operatorname{prem}(G, F)$.

The following theorem, whose proof can be found in Ritt (1950), II, §13, gives an algorithmic method to solve the ideal membership problem of $\{F\} : S$.

Theorem 2.1.1. *For every $G \in \mathbb{K}(x)\{y\}$, $G \in \{F\} : S \Longleftrightarrow \operatorname{prem}(G, F) = 0$.*

Corollary 2.1.2. *Suppose that η is a general solution of $F(y) = 0$. Then for every $G \in \mathbb{K}(x)\{y\}$, $G(\eta) = 0 \Longleftrightarrow \operatorname{prem}(G, F) = 0$.*

Observe that $S \notin \{F\} : S$ because $\operatorname{prem}(S, F) = S \neq 0$. Therefore, a general solution of $F(y) = 0$ is not annulled by S.

It is known from the point of view of analysis[†] that the most general solution of $F(x, y, y')$ contains one arbitrary constant (e.g. Ince (1926); Piaggio (1933))[‡]. The conclusion applies for higher order ODEs accordingly. In differential algebra context, one have to make the meaning of the term "arbitrary constant" precisely. By Ritt (1950), III, §5, an *arbitrary constant* w.r.t a given field $\mathbb{K}$ is a quantity c which can be adjoined to the field $\mathbb{K}$—to obtain an extension field $\mathbb{K}(c)$—which is transcendental over $\mathbb{K}$ and the derivative of c in the extension field $\mathbb{K}(c)$ is zero.

Let us see the above fact in the case of rational general solutions. Suppose that

$$y^* = \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0}$$

is a rational general solution of $F(y) = 0$. Then there exists a coefficient of y^* does not belong to $\mathbb{K}$. Otherwise, the differential polynomial (of order 0)

$$G = (b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0)y - (a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0) \in \mathbb{K}(x)\{y\}$$

vanishes on y^*, but $\operatorname{prem}(G, F) = G \neq 0$. Therefore, y^* must contain a constant which is not in $\mathbb{K}$ and hence it is transcendental over $\mathbb{K}$ because $\mathbb{K}$ is algebraically closed.

[†]A solution is expanded into a Taylor series, one looks for its coefficients in the expansion.
[‡]In Piaggio (1933): Chapter I, §7.

Remark 2.1.1. Suppose that $y = f(x, c)$ is a rational general solution of $F(x, y, y') = 0$, where c is an arbitrary constant, i.e., we have

$$F(x, f(x, c), f_x(x, c)) = 0,$$

where f_x is the partial derivative of f w.r.t. x. Let us view x and c as parameters of the rational map

$$\mathcal{P}(x, c) = (x, f(x, c), f_x(x, c)). \tag{2.4}$$

The Jacobian matrix of $\mathcal{P}(x, c)$ is

$$J_{\mathcal{P}} = \begin{pmatrix} 1 & f_x(x, c) & f_{xx}(x, c) \\ 0 & f_c(x, c) & f_{xc}(x, c) \end{pmatrix}. \tag{2.5}$$

Since f effectively depends on c, we have $f_c(x, c) \neq 0$. Therefore, the generic rank of $J_{\mathcal{P}}$ is 2. Hence, $\mathcal{P}(x, c)$ is a rational parametrization of the surface $F(x, y, z) = 0$. This is the reason why we restrict the consideration to the class of rational parametrizable algebraic ODEs of order 1.

2.2 The associated system of first-order parametrizable algebraic ODEs

2.2.1 Determination of the associated system

In this section, we present a method to determine a rational general solution of the first-order algebraic ODE

$$F(x, y, y') = 0, \tag{2.6}$$

where $F(x, y, z)$, an irreducible polynomial in $\mathbb{K}[x, y, z]$, defines a rational surface.

First of all, suppose that $y = f(x)$ is a rational solution of $F(x, y, y') = 0$. Then it generates a rational space curve parametrized by $(x, f(x), f'(x))$, here x is viewed as a parameter, and the curve belongs to the algebraic surface defined by $F(x, y, z)$. Therefore, solving the differential equation $F(x, y, y') = 0$ amounts to look for all such parametric curves on the algebraic surface $F(x, y, z) = 0$. By Remark 2.1.1, it is natural to consider those algebraic surfaces possessing rational parametrizations.

Definition 2.2.1. The algebraic surface $F(x, y, z) = 0$ is called a *unirational surface* iff there exists a rational map

$$\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t)) \tag{2.7}$$

such that $F(\mathcal{P}(s,t)) = 0$, where χ_1, χ_2 and χ_3 are rational functions in s and t, and the Jacobian matrix of $\mathcal{P}(s,t)$ has a generic rank 2. Then $\mathcal{P}(s,t)$ is called a *rational parametrization* of $F(x,y,z) = 0$.

Since the Jacobian matrix $J_{\mathcal{P}}$ of $\mathcal{P}(s,t)$ has a generic rank 2, at least one of the 2×2 minors of $J_{\mathcal{P}}$ is non-zero. We can assume w.l.o.g that

$$\chi_{1s}\chi_{2t} - \chi_{1t}\chi_{2s} \neq 0, \tag{2.8}$$

where χ_{is}, χ_{it} are the partial derivatives of χ_i w.r.t s and t. Because if it is not the case, then $\chi_1(s,t)$ and $\chi_2(s,t)$ must be related by the equation $\chi_1(s,t) = \phi(-\chi_2(s,t))$, where $\phi(y)$ is an arbitrary function of one variable. Since χ_1 and χ_2 are rational functions, it follows that $\phi(y)$ is a rational function. Then the surface $x - \phi(-y) = 0$ is a component of $F(x,y,z) = 0$, hence by irreducibility, $F(x,y,z)$ is the numerator of $x - \phi(-y)$. In this case, the surface defined by F is not corresponding to any first-order algebraic ODE.

Definition 2.2.2. A rational parametrization $\mathcal{P}(s,t)$ of $F(x,y,z) = 0$ is called *proper* iff it has an inverse and its inverse is also rational, i.e., there is a rational map

$$\mathcal{Q}(x,y,z) = (\psi_1(x,y,z), \psi_2(x,y,z))$$

such that $(\mathcal{Q} \circ \mathcal{P})(s,t) = (s,t)$ for almost all s, t and $(\mathcal{P} \circ \mathcal{Q})(x,y,z) = (x,y,z)$ for almost all (x,y,z) on the surface $F(x,y,z) = 0$ or, equivalently, $\mathbb{K}(\mathcal{P}(s,t)) = \mathbb{K}(s,t)$. Such a $\mathcal{P}(s,t)$ is called a birational map. The surface defined by $F(x,y,z) = 0$ is called *rational* iff it has a proper rational parametrization.

Note that, the parametrization (2.4) in Remark 2.1.1 may be not proper.

Definition 2.2.3. The *solution surface* of $F(x,y,y') = 0$, denoted by $\mathcal{S}$, is the surface $F(x,y,z) = 0$ when we view x, y, z as independent variables.

Definition 2.2.4. An algebraic ODE $F(x,y,y') = 0$ is called a *parametrizable algebraic ODE* if its solution surface is rational, i.e., it admits a rational parametrization of the form (2.7).

In the sequel, we denote by $\mathcal{AODE}$ the set $\mathcal{AODE} = \{F(x,y,y') = 0 \mid F \in \mathbb{K}[x,y,z]\}$ and by $\mathcal{PODE}$ the set

$$\mathcal{PODE} = \{F \in \mathcal{AODE} \mid \text{the solution surface } F = 0 \text{ is rationally parametrizable}\}.$$

In $\mathcal{AODE}$, if F is not involving x, then the differential equation (2.6) is called *autonomous*. In general, F is possibly involving x, the differential equation (2.6) is called *non-autonomous*. We view an autonomous ODE as a special case of the non-autonomous one.

Definition 2.2.5. Let $y = f(x)$ be a rational solution of $F(x, y, y') = 0$. The space curve parametrized by $\mathcal{C}(x) = \{(x, f(x), f'(x)) \mid x \in \mathbb{K}\}$ is called the *solution curve* of f w.r.t. $F(x, y, y') = 0$ or simply a solution curve of f when the differential equation is clear from the context.

In some textbooks, the curve $\mathcal{C}(x) = (x, f(x), f'(x))$ is called an *integral curve*. We will use this terminology when we consider the curve $\mathcal{C}(x)$ without taking into account an algebraic differential equation $F(x, y, y') = 0$ having $f(x)$ as a solution. If $f(x)$ is a rational function in x, then one can easily generate an algebraic differential equation $F(x, y, y') = 0$ having $f(x)$ as a solution. Hence, $\mathcal{C}(x)$ becomes a solution curve of f w.r.t. $F(x, y, y') = 0$.

From now on, we always consider F in $\mathcal{PODE}$ and $\mathcal{P}(s, t)$ to be a proper rational parametrization of $F(x, y, z) = 0$. The inverse map of $\mathcal{P}$, denoted by $\mathcal{P}^{-1}$, defines on the surface $\mathcal{S}$, except for finitely many curves or points on $\mathcal{S}$.

Definition 2.2.6. Let $f(x)$ be a rational solution of the equation $F(x, y, y') = 0$. Let $\mathcal{S}$ be the solution surface of $F(x, y, y') = 0$ and $\mathcal{C}(x)$ be the solution curve of f. Let $\mathcal{P}$ be a proper rational parametrization of $F(x, y, z) = 0$. The solution curve $\mathcal{C}(x)$ is *parametrizable* by $\mathcal{P}$ iff $\mathcal{C}(x)$ is almost contained in $\mathrm{im}(\mathcal{P}) \cap \mathrm{dom}(\mathcal{P}^{-1})$, i.e., except for finitely many points on $\mathcal{C}(x)$. Here $\mathrm{im}(\mathcal{P})$ and $\mathrm{dom}(\mathcal{P}^{-1})$ are the image and the domain of the corresponding maps.

Proposition 2.2.1. *Let $F \in \mathcal{PODE}$ with a proper parametrization $\mathcal{P}(s, t)$. The differential equation $F(x, y, y') = 0$ has a rational solution whose solution curve is parametrizable by $\mathcal{P}$ if and only if the system*

$$\begin{cases} \chi_1(s(x), t(x)) = x \\ \chi_2(s(x), t(x))' = \chi_3(s(x), t(x)) \end{cases} \tag{2.9}$$

has a rational solution $(s(x), t(x))$. In that case, $y = \chi_2(s(x), t(x))$ is a rational solution of $F(x, y, y') = 0$.

Proof. Assume that $y = f(x)$ is a rational solution of $F(x, y, y') = 0$ and the solution curve of $f(x)$ is parametrizable by $\mathcal{P}$. Let $(s(x), t(x)) = \mathcal{P}^{-1}(x, f(x), f'(x))$. Then $s(x)$ and $t(x)$ are rational functions because $f(x)$ is a rational function and $\mathcal{P}^{-1}$ is a rational map. We have

$$\mathcal{P}(s(x), t(x)) = \mathcal{P}(\mathcal{P}^{-1}(x, f(x), f'(x))) = (x, f(x), f'(x)).$$

In other words, $(s(x), t(x))$ is a rational solution of the system

$$\begin{cases} \chi_1(s(x), t(x)) = x \\ \chi_2(s(x), t(x)) = f(x) \\ \chi_3(s(x), t(x)) = f'(x). \end{cases} \tag{2.10}$$

Therefore, $\chi_1(s(x), t(x)) = x$ and $\chi_2(s(x), t(x))' = \chi_3(s(x), t(x))$. Conversely, if two rational functions $s = s(x)$ and $t = t(x)$ satisfy the system (2.9), then $y = \chi_2(s(x), t(x))$ is a rational solution of $F(x, y, y') = 0$ because $F(\mathcal{P}(s(x), t(x))) = 0$. □

Note that most of solution curves of $F(x, y, y') = 0$ will be parametrizable by $\mathcal{P}$ because $\mathcal{P}$ covers almost all the solution surface $\mathcal{S}$ and $\mathcal{P}^{-1}$ is defined at almost everywhere on the solution surface $\mathcal{S}$ except for finitely many curves or points on $\mathcal{S}$.

Q 2. *What are rational solutions of the system* (2.9)*?*

Let us have a closed looking at the system (2.9). We see that it can be decomposed as a differential system and an algebraic system. Indeed, differentiating the first equation of (2.9) and expanding the last equation of (2.9), we obtain a linear system of equations in $s'(x)$ and $t'(x)$

$$\begin{cases} \dfrac{\partial \chi_1(s(x), t(x))}{\partial s} \cdot s'(x) + \dfrac{\partial \chi_1(s(x), t(x))}{\partial t} \cdot t'(x) = 1 \\ \\ \dfrac{\partial \chi_2(s(x), t(x))}{\partial s} \cdot s'(x) + \dfrac{\partial \chi_2(s(x), t(x))}{\partial t} \cdot t'(x) = \chi_3(s(x), t(x)). \end{cases} \tag{2.11}$$

If

$$\det \begin{pmatrix} \dfrac{\partial \chi_1(s(x), t(x))}{\partial s} & \dfrac{\partial \chi_1(s(x), t(x))}{\partial t} \\ \\ \dfrac{\partial \chi_2(s(x), t(x))}{\partial s} & \dfrac{\partial \chi_2(s(x), t(x))}{\partial t} \end{pmatrix} \neq 0, \tag{2.12}$$

then $(s(x), t(x))$ is a rational solution of the autonomous system of differential equations

$$\left\{ s' = \frac{f_1(s, t)}{g(s, t)},\ \ t' = \frac{f_2(s, t)}{g(s, t)} \right\} \tag{2.13}$$

where $f_1(s, t), f_2(s, t), g(s, t) \in \mathbb{K}(s, t)$ are defined by

$$\begin{aligned} f_1(s, t) &= \frac{\partial \chi_2(s, t)}{\partial t} - \chi_3(s, t) \cdot \frac{\partial \chi_1(s, t)}{\partial t}, \quad f_2(s, t) = \chi_3(s, t) \cdot \frac{\partial \chi_1(s, t)}{\partial s} - \frac{\partial \chi_2(s, t)}{\partial s}, \\ g(s, t) &= \frac{\partial \chi_1(s, t)}{\partial s} \cdot \frac{\partial \chi_2(s, t)}{\partial t} - \frac{\partial \chi_1(s, t)}{\partial t} \cdot \frac{\partial \chi_2(s, t)}{\partial s}. \end{aligned} \tag{2.14}$$

If the determinant (2.12) is equal to 0, then $(s(x), t(x))$ is a solution of the algebraic system

$$\left\{\bar{g}(s,t) = 0, \;\; \bar{f}_1(s,t) = 0\right\} \tag{2.15}$$

where $\bar{g}(s,t)$ and $\bar{f}_1(s,t)$ are the numerators of $g(s,t)$ and $f_1(s,t)$, respectively. In the latter case, $(s(x), t(x))$ defines a curve if and only if $\gcd(\bar{g}(s,t), \bar{f}_1(s,t))$ is a non-constant polynomial in s, t. Otherwise, $(s(x), t(x))$ is just an intersection point of the two algebraic curves $\bar{g}(s,t) = 0$ and $\bar{f}_1(s,t) = 0$, which does not satisfy the relation (2.9).

Therefore, the rational solutions of the system (2.9) is the union of the rational solutions of (2.13) and the non-trivial rational solutions of (2.15).

Definition 2.2.7. The autonomous system (2.13) is called the *associated system* of the differential equation $F(x, y, y') = 0$ with respect to $\mathcal{P}(s,t)$.

The main features of the associated system are autonomous, of order 1 and of degree 1 with respect to s' and t'. Later, these features turn out to be the advantages of the approach.

Claim 1. *A rational general solution of the system* (2.13) *completely determines a rational general solution of the differential equation* $F(x, y, y') = 0$.

At this point we define, from the point of view of differential algebra, what we mean by a rational general solution of the system (2.13). For this purpose we need some preparations.

2.2.2 Rational general solutions of the associated system

Consider the new differential ring $\mathbb{K}(x)\{s,t\}$ with the usual derivation δ extended from the derivation $\dfrac{d}{dx}$ of $\mathbb{K}(x)$, where s, t are two differential indeterminates. We denote by s_i and t_i the i-th derivatives of s and t, respectively.

Definition 2.2.8. Let $\mathcal{V} = \{s_i \mid i \in \mathbb{N}\} \cup \{t_i \mid i \in \mathbb{N}\}$. The *ord-lex ranking* on $\mathcal{V}$ is the total order defined as follows:

$$\begin{cases} s_i < s_j \text{ if } i < j, \\ t_i < t_j \text{ if } i < j, \\ t_i < s_j \text{ if } i \leq j, \\ s_i < t_j \text{ if } i < j. \end{cases}$$

The ord-lex ranking is an orderly ranking (see the appendix B.2). We use this ranking on the differential ring $\mathbb{K}(x)\{s,t\}$ from now on.

Definition 2.2.9. Let $F, G \in \mathbb{K}(x)\{s,t\}$. F is said to be *of higher rank than G in s* iff one of the following conditions holds:

1. $\operatorname{ord}_s(F) > \operatorname{ord}_s(G)$;

2. $\operatorname{ord}_s(F) = \operatorname{ord}_s(G) = n$ and $\deg_{s_n}(F) > \deg_{s_n}(G)$.

If F is of higher rank than G in s, then we also say G is *of lower rank than F in s*. Analogously these notions are defined for t.

Definition 2.2.10. Let F be a differential polynomial in $\mathbb{K}(x)\{s,t\}$. The *leader* of F is the highest derivative occurring in F with respect to the ord-lex ranking on the set of derivatives $\mathcal{V}$. The *initial* of F is the leading coefficient of F with respect to its leader. The *separant* of F is the partial derivative of F with respect to its leader.

Observe that the separant of F is also the initial of any proper derivative $\delta^i(F)$ of F.

Definition 2.2.11. Let F and G be differential polynomials in $\mathbb{K}(x)\{s,t\}$. G is said to be *reduced* with respect to F iff G is of lower rank than F in the indeterminate defining the leader of F.

Let $\mathcal{A}$ be an autoreduced set in the differential ring $\mathcal{R} = \mathbb{K}(x)\{s,t\}$. Let $G \in \mathbb{K}(x)\{s,t\}$. By Ritt's reduction, Ritt (1950), Kolchin (1973)[§], there exist $R \in \mathcal{R}$, $s_A, i_A \in \mathbb{N}$ such that R is reduced w.r.t. $\mathcal{A}$, the rank of R is lower than or equal to that of G and

$$\prod_{A\in\mathcal{A}} I_A^{i_A} S_A^{s_A} G - R$$

can be written as a linear combination over $\mathcal{R}$ of derivatives $\{\delta^i(A) \mid A \in \mathcal{A}, \delta^i(u_A) \le u_G\}$, where u_A and u_G are the leader of A and G, respectively. The differential polynomial R is called the *differential pseudo remainder* of G with respect to $\mathcal{A}$, denoted by

$$R = \operatorname{prem}(G, \mathcal{A}).$$

From now on, we consider $M_i, N_i \in \mathbb{K}[s,t]$, $N_i \neq 0$, $\gcd(M_i, N_i) = 1$ for $i = 1, 2$ and two special differential polynomials F_1 and F_2 in $\mathcal{R}$ defined as follows

$$F_1 := N_1 s' - M_1, \quad F_2 := N_2 t' - M_2.$$

In fact, F_1 and F_2 are in the subring $\mathbb{K}\{s,t\}$ of autonomous differential polynomials of $\mathcal{R}$. The leaders of F_1 and F_2 are s' and t', respectively. Moreover, $\deg_{s'}(F_1) = \deg_{t'}(F_2) = 1$. It follows that the initial and separant of F_1 (respectively, of F_2) are the same. The differential ideal generated by F_1 and F_2 is denoted by $[F_1, F_2]$. In applications, we will take M_1, M_2, N_1, N_2 to be the polynomials in the numerators and the denominators of the right hand side of the associated system (2.13).

[§]Kolchin (1973), I, Proposition 1.

The set $\mathcal{A} = \{F_1, F_2\}$ is an *autoreduced set* relative to the ord-lex ranking because F_1 is reduced with respect to F_2 and F_2 is reduced with respect to F_1. Now by Ritt's reduction, for any $G \in \mathbb{K}(x)\{s,t\}$, we can reduce G w.r.t. the autoreduced set $\mathcal{A}$ to obtain the differential pseudo remainder $R = \text{prem}(G, \mathcal{A})$, which is also denoted by $R = \text{prem}(G, F_1, F_2)$ in an explicit form.

Observation 2.2.1. *Observe that the differential pseudo remainder $R = prem(G, F_1, F_2)$ is always a polynomial in $\mathbb{K}(x)[s,t]$ because F_1 and F_2 are of order 1 and of degree 1 w.r.t. their leaders.*

Lemma 2.2.2. *Let*

$$\mathcal{I} = \{G \in \mathbb{K}(x)\{s,t\} \mid \text{prem}(G, F_1, F_2) = 0\}.$$

Then $\mathcal{I}$ is a prime differential ideal in $\mathbb{K}(x)\{s,t\}$.

Proof. Let $H_{\mathcal{A}} = N_1 N_2$ and denote $H_{\mathcal{A}}^{\infty} = \{N_1^{m_1} N_2^{m_2} \mid m_1, m_2 \in \mathbb{N}\}$. Then

$$[F_1, F_2] : H_{\mathcal{A}}^{\infty} := \{G \in \mathbb{K}(x)\{s,t\} \mid \exists J \in H_{\mathcal{A}}^{\infty}, JG \in [F_1, F_2]\}$$

is a prime differential ideal (Ritt (1950), V, §3, page 107). We prove that

$$\mathcal{I} = [F_1, F_2] : H_{\mathcal{A}}^{\infty}.$$

In fact, it is clear that $\mathcal{I} \subseteq [F_1, F_2] : H_{\mathcal{A}}^{\infty}$. Let $G \in [F_1, F_2] : H_{\mathcal{A}}^{\infty}$. Then there exists $J \in H_{\mathcal{A}}^{\infty}$ such that $JG \in [F_1, F_2]$. On the other hand, let $R = \text{prem}(G, F_1, F_2)$, we have

$$J_1 G - R \in [F_1, F_2]$$

for some $J_1 \in H_{\mathcal{A}}^{\infty}$. It follows that $JR \in [F_1, F_2]$. Since R and J are in $\mathbb{K}(x)[s,t]$, we have $JR \in [F_1, F_2]$ if and only if $JR = 0$. We must have $R = 0$ because $J \neq 0$. Therefore, $\mathcal{I} = [F_1, F_2] : H_{\mathcal{A}}^{\infty}$, i.e., $\mathcal{I}$ is a prime differential ideal. □

Definition 2.2.12. Let $M_i, N_i \in \mathbb{K}[s,t]$, $N_i \neq 0$, $\gcd(M_i, N_i) = 1$ for $i = 1, 2$. A rational solution $(s(x), t(x))$ of the autonomous system

$$\left\{ s' = \frac{M_1(s,t)}{N_1(s,t)},\ t' = \frac{M_2(s,t)}{N_2(s,t)} \right\} \tag{2.16}$$

is called a *rational general solution* iff it is a rational generic zero of the prime differential ideal $\mathcal{I}$, i.e., for any $G \in \mathbb{K}(x)\{s,t\}$, we have

$$G(s(x), t(x)) = 0 \Longleftrightarrow \text{prem}(G, N_1 s' - M_1, N_2 t' - M_2) = 0. \tag{2.17}$$

Lemma 2.2.3. *Let* $(s(x), t(x))$ *be a rational general solution of the system* (2.16)*. Let* G *be a bivariate polynomial in* $\mathbb{K}(x)[s,t]$*. Then* $G(s(x), t(x)) = 0 \Longleftrightarrow G = 0$ *in* $\mathbb{K}(x)[s,t]$.

Proof. Since $G \in \mathbb{K}(x)[s,t]$, we have $\mathrm{prem}(G, N_1 s' - M_1, N_2 t' - M_2) = G$. By definition of general solutions, $G(s(x), t(x)) = 0 \Longleftrightarrow G = 0$ in $\mathbb{K}(x)[s,t]$. □

Lemma 2.2.4. *Let*

$$s(x) = \frac{a_k x^k + a_{k-1}x^{k-1} + \cdots + a_0}{b_l x^l + b_{l-1}x^{l-1} + \cdots + b_0} \quad \textit{and} \quad t(x) = \frac{c_m x^m + c_{m-1}x^{m-1} + \cdots + c_0}{d_n x^n + d_{n-1}x^{n-1} + \cdots + d_0}$$

be a non-trivial rational solution of the system (2.16)*, where* a_i, b_i, c_i, d_i *are in some field of constants* $\mathbb{L}$*, extended from* $\mathbb{K}$*; and* $b_l, d_n \neq 0$*. If* $(s(x), t(x))$ *is a rational general solution of the system* (2.16)*, then there exists a constant, which is transcendental over* $\mathbb{K}$*, among the coefficients of* $s(x)$ *and* $t(x)$.

Proof. Let $S = (b_l x^l + b_{l-1}x^{l-1} + \cdots + b_0)s - (a_k x^k + a_{k-1}x^{k-1} + \cdots + a_0)$ and $T = (d_n x^n + d_{n-1}x^{n-1} + \cdots + d_0)t - (c_m x^m + c_{m-1}x^{m-1} + \cdots + c_0)$. Let $G = \mathrm{res}_x(S, T)$ be the resultant of S and T with respect to x. Then G is a polynomial in s and t with the coefficients depending on a_i, b_i, c_i, d_i. If all a_i, b_i, c_i, d_i were in $\mathbb{K}$, then $G \in \mathbb{K}[s,t]$ and $G(s(x), t(x)) = 0$. Since $(s(x), t(x))$ is a rational general solution, it follows by Lemma 2.2.3 that $G = 0$. But G is the implicit equation of the rational curve with parametrization $(s(x), t(x))$; compare Chapter 4 §4.5 in Sendra et al. (2008). So $G \neq 0$, in contradiction. Therefore, there is a coefficient of $s(x)$ or $t(x)$ that does not belong to $\mathbb{K}$. Since $\mathbb{K}$ is an algebraically closed field, a constant which is not in $\mathbb{K}$ must be a transcendental element over $\mathbb{K}$. □

This lemma gives us a necessary condition for $(s(x), t(x))$ to be a rational general solution of the system (2.16). It requires that any rational general solution of the system has to contain at least one coefficient transcendental over the constant field of the system itself. As an early discussion in the chapter, this transcendental coefficient is an arbitrary constant. Next we give a sufficient condition for a rational solution $(s(x), t(x))$ of the system (2.16) to be a rational general solution.

Lemma 2.2.5. *Let* $(s(x), t(x))$ *be a rational solution of the system* (2.16)*. Let* $H(s,t)$ *be the monic defining polynomial (w.r.t. a lexicographic order of terms in* s *and* t*) of the rational algebraic curve defined by* $(s(x), t(x))$*. If there is an arbitrary constant in the set of coefficients of* $H(s,t)$*, then* $(s(x), t(x))$ *is a rational general solution of the system* (2.16)*.*

Proof. Suppose that $(s(x), t(x))$ is a rational solution of the system (2.16). Let $G \in$

$\mathbb{K}(x)\{s, t\}$ be a differential polynomial such that $G(s(x), t(x)) = 0$. Let

$$R = \text{prem}(G, N_1 s' - M_1, N_2 t' - M_2).$$

Then $R \in \mathbb{K}(x)[s, t]$ and $R(s(x), t(x)) = 0$. It is sufficient to assume that there is only one arbitrary constant c in the set of coefficients of $H(s, t)$. Then $H(s, t) \in \mathbb{K}(c)[s, t]$. Moreover, $H(s, t)$ is irreducible over $\overline{\mathbb{K}(c)}$ because it is a rational curve (Sendra et al. (2008), Theorem 4.4). Therefore, the polynomial $R(s, t)$ must be a multiple of $H(s, t)$. This happens if and only if $R = 0$. It follows that $(s(x), t(x))$ is a rational general solution of the system (2.16). □

> **Q 3.** *What is the inverse image of a solution curve of a rational general solution of $F(x, y, y') = 0$?*

Theorem 2.2.6. *Let $y = f(x)$ be a rational general solution of $F(x, y, y') = 0$. Suppose that the solution curve of f is parametrizable by $\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t))$. Let*

$$(s(x), t(x)) = \mathcal{P}^{-1}(x, f(x), f'(x))$$

and $g(s, t) = \dfrac{\partial \chi_1(s, t)}{\partial s} \cdot \dfrac{\partial \chi_2(s, t)}{\partial t} - \dfrac{\partial \chi_1(s, t)}{\partial t} \cdot \dfrac{\partial \chi_2(s, t)}{\partial s}$. Then $g(s(x), t(x)) \neq 0$ and $(s(x), t(x))$ is a rational general solution of the system (2.13).

Proof. It is sufficient to prove the claim that if $R \in \mathbb{K}(x)[s, t]$ and $R(s(x), t(x)) = 0$, then $R = 0$. If this is done, then $g(s(x), t(x)) \neq 0$ because $g(s, t) \neq 0$ by (2.8). Suppose that $P \in \mathbb{K}(x)\{s, t\}$ is a differential polynomial such that $P(s(x), t(x)) = 0$. Let

$$R = \text{prem}(P, N_1 s' - M_1, N_2 t' - M_2),$$

where M_1, M_2, N_1, N_2 are numerators and denominators of the right hand side of the system (2.13). Then $R \in \mathbb{K}(x)[s, t]$ and $P(s(x), t(x)) = 0$ implies that $R(s(x), t(x)) = 0$. By the claim, $R = 0$. Hence $(s(x), t(x))$ is a rational general solution of the system (2.13).

Now it remains to prove the claim. We have

$$R(s(x), t(x)) = R(\mathcal{P}^{-1}(x, f(x), f'(x))) = 0.$$

Let us consider the rational function $R(\mathcal{P}^{-1}(x, y, z)) = \dfrac{U(x, y, z)}{V(x, y, z)}$. Then $U(x, y, y')$ is a differential polynomial satisfying the condition

$$U(x, f(x), f'(x)) = 0.$$

Since $f(x)$ is a rational general solution of $F(y) = 0$ and $U(x, y, y')$ vanishes on $y = f(x)$, the differential pseudo remainder of U with respect to F must be zero. On the other hand, both F and U are differential polynomials of order 1, we only divide U by F and not by any of its derivatives. Hence, we have the reduction

$$I^m U = Q_0 F,$$

where I is the initial of F, $m \in \mathbb{N}$ and Q_0 is a differential polynomial of order at most 1 in $\mathbb{K}(x)\{y\}$. Therefore,

$$R(s,t) = R(\mathcal{P}^{-1}(\mathcal{P}(s,t))) = \frac{U(\mathcal{P}(s,t))}{V(\mathcal{P}(s,t))} = \frac{Q_0(\mathcal{P}(s,t))F(\mathcal{P}(s,t))}{I^m(\mathcal{P}(s,t))V(\mathcal{P}(s,t))} = 0$$

because $F(\mathcal{P}(s,t)) = 0$ and $I(\mathcal{P}(s,t)) \neq 0$. □

Q 4. *Suppose that we have a rational solution of $F(x,y,y') = 0$ but it is not parametrizable by $\mathcal{P}$? Could this solution be a rational general solution of $F(x,y,y') = 0$?*

Proposition 2.2.7. *If $y = f(x)$ is a rational solution of $F(x,y,y') = 0$ but it is not parametrizable by $\mathcal{P}$, then it can not be a rational general solution of $F(x,y,y') = 0$.*

Proof. Since $y = f(x)$ is not parametrizable by $\mathcal{P}$, the solution curve $(x, f(x), f'(x))$ must lie on the intersection of the solution surface $F(x,y,z) = 0$ and another surface $G(x,y,z) = 0$ defined by the denominators of the inverse map $\mathcal{P}^{-1}$. The resultant $R(x,y) = res_z(F,G)$ is vanished on $f(x)$ and reduced w.r.t. F. By definition, this can not be a general solution. □

Q 5. *How to construct a rational general solution of $F(x,y,y') = 0$ from a rational general solution of its associated system?*

Assume that $(s(x), t(x))$ is a rational general solution of the associated system (2.13). Substituting $s(x)$ and $t(x)$ into $\chi_1(s,t)$ and using the relation (2.11) we get $\chi_1(s(x), t(x)) = x + c$ for some constant c. Hence $\chi_1(s(x-c), t(x-c)) = x$. It follows that

$$y = \chi_2(s(x-c), t(x-c)) \tag{2.18}$$

is a solution of the differential equation $F(x,y,y') = 0$. Moreover, we will prove that $y = \chi_2(s(x-c), t(x-c))$ is a rational general solution of $F(x,y,y') = 0$.

Theorem 2.2.8. *Let $(s(x), t(x))$ be a rational general solution of the system* (2.13). *Let $c = \chi_1(s(x), t(x)) - x$. Then $y = \chi_2(s(x-c), t(x-c))$ is a rational general solution of $F(x,y,y') = 0$.*

Proof. By the above discussion, it is clear that $y = \chi_2(s(x-c), t(x-c))$ is a rational solution of $F(x,y,y') = 0$. Let G be an arbitrary differential polynomial in $\mathbb{K}(x)\{y\}$ such that $G(y) = 0$. Let $R = \mathrm{prem}(G, F)$ be the differential pseudo remainder of G with respect to F. It follows that $R(y) = 0$. We have to prove that $R = 0$. Assume that $R \neq 0$. Then

$$R(\chi_1(s,t), \chi_2(s,t), \chi_3(s,t)) = \frac{W(s,t)}{Z(s,t)} \in \mathbb{K}(s,t).$$

On the other hand,

$$R(\mathcal{P}(s(x-c), t(x-c))) = R(x, y, y') = 0.$$

It follows that $W(s(x-c), t(x-c)) = 0$, hence, $W(s(x), t(x)) = 0$. By Lemma 2.2.3 we must have $W(s,t) = 0$. Thus $R(\chi_1(s,t), \chi_2(s,t), \chi_3(s,t)) = 0$. Since F is irreducible and $\deg_{y'} R < \deg_{y'} F$, we have $R = 0$ in $\mathbb{K}[x,y,z]$. Therefore, y is a rational general solution of $F(x,y,y') = 0$. □

2.2.3 Algorithm and example

Theorem 2.2.6 and Theorem 2.2.8 give a method to determine a rational general solution of the first-order parametrizable algebraic ODE $F(x,y,y') = 0$. We summarize the procedure by the following semi-algorithm. It depends on a method for solving the system (2.13).

Algorithm `GENERALSOLVER`
Input: $F(x,y,y') = 0$ and $\mathcal{P}(s,t) = (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t))$ such that $F(\mathcal{P}(s,t)) = 0$ and $\mathcal{P}(s,t)$ is proper, where $F \in \mathbb{K}[x,y,z]$ and $\chi_1, \chi_2, \chi_3 \in \mathbb{K}(s,t)$.
Output: a rational general solution of $F(x,y,y') = 0$ in the affirmative case.

1. Compute $f_1(s,t), f_2(s,t), g(s,t)$ as in (2.14).
2. Compute a rational general solution $(s(x), t(x))$ of the associated system
$$\left\{ s' = \frac{f_1(s,t)}{g(s,t)}, \quad t' = \frac{f_2(s,t)}{g(s,t)} \right\}.$$
3. Compute the constant $c := \chi_1(s(x), t(x)) - x$.
4. Return $y = \chi_2(s(x-c), t(x-c))$.

Note that we still have to solve the associated system for its rational general solutions in general cases. The rest of this chapter and the next chapters will develop a method for determining a rational general solution of such systems.

Example 2.2.1. Consider the differential equation

$$F(x, y, y') \equiv y'^3 - 4xyy' + 8y^2 = 0. \tag{2.19}$$

Note that this differential equation appears in Piaggio (1933), Chapter XV, Arts 161, in Hubert (1996) and in Kamke (1948), equation I.525. Here we demonstrate our approach for this differential equation. The solution surface $z^3 - 4xyz + 8y^2 = 0$ has a proper parametrization¶

$$\mathcal{P}(s,t) = (t, -4s^2 \cdot (2s-t), -4s \cdot (2s-t)).$$

The inverse map is $\mathcal{P}^{-1}(x,y,z) = \left(\frac{y}{z}, x\right)$. We compute

$$g(s,t) = 8s \cdot (3s-t),$$

$$f_1(s,t) = 4s \cdot (3s-t), \quad f_2(s,t) = 8s \cdot (3s-t).$$

In this case, the associated system is very simple $\left\{s' = \frac{1}{2}, \ t' = 1\right\}$. Solving this system we obtain a rational general solution $s(x) = \frac{x}{2} + c_2$, $t(x) = x + c_1$ for arbitrary constants c_1, c_2. The above algorithm follows that the rational general solution of the differential equation $F(x, y, y') = 0$ is

$$y = -4s(x-c_1)^2 \cdot (2s(x-c_1) - t(x-c_1)) = -c(x+c)^2, \tag{2.20}$$

where $c = 2c_2 - c_1$. The following is the graph of three solution curves

$$(x, -c(x+c)^2, -2c(x+c))$$

with $c = -1, 1$ and 2 on the solution surface $z^3 - 4xyz + 8y^2 = 0$.

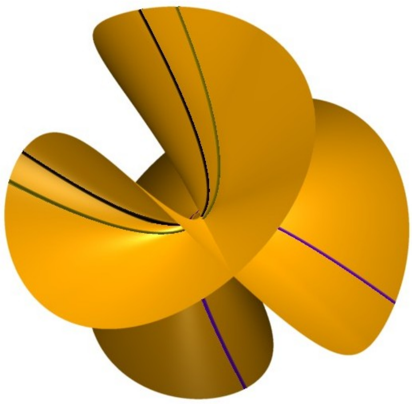

Figure 2.1: Some solution curves on the solution surface $z^3 - 4xyz + 8y^2 = 0$

¶It can be seen that the cubic curve $z^3 - 4xyz + 8y^2 = 0$ is rational over $\mathbb{K}(x)$.

Note that, in this example, $\gcd(g(s,t), f_1(s,t)) = 4s \cdot (3s - t)$. This defines a reducible algebraic curve and we still find some solutions of the differential equation (2.19), whose solution curves are parametrizable by $\mathcal{P}(s,t)$, by solving the system

$$\begin{cases} t(x) = x \\ -4s(x)^2 \cdot (2s(x) - t(x)) = f(x) \\ -4s(x) \cdot (2s(x) - t(x)) = f'(x) \\ 4s(x) \cdot (3s(x) - t(x)) = 0. \end{cases} \tag{2.21}$$

This system has two different solutions, namely

$$(s(x), t(x)) = (0, x) \quad \text{and} \quad (s(x), t(x)) = \left(\frac{x}{3}, x\right).$$

These solutions give us two other solutions of the equation (2.19), namely $y = 0$ and $y = \frac{4}{27}x^3$. The solution $y = 0$ can be obtained by specifying the constant $c = 0$ in the general solution (2.20). However, we can not get the solution $y = \frac{4}{27}x^3$ from the general solution (2.20). Note that the separant of F is $S = 3y'^2 - 4xy$. We can prove that the common solutions of F and S, which are called *singular solutions* of $F(x, y, y') = 0$, are only $y = 0$ and $y = \frac{4}{27}x^3$. Here is the graph of the cubic curve $\left(x, \frac{4}{27}x^3, \frac{4}{9}x^2\right)$ generated by the singular solution $y = \frac{4}{27}x^3$. ‖

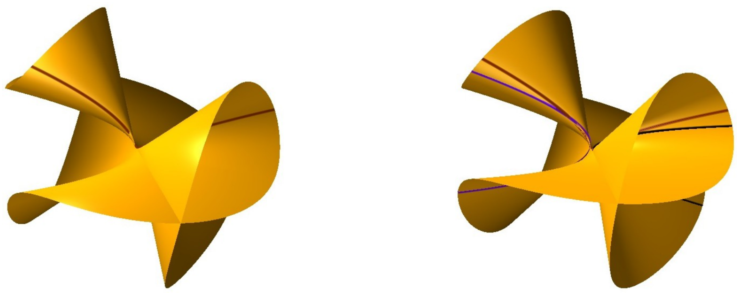

Figure 2.2: The cubic curve $\left(x, \frac{4}{27}x^3, \frac{4}{9}x^2\right)$ on the solution surface $z^3 - 4xyz + 8y^2 = 0$

Remark 2.2.1. Let $F(x, y, y') = 0$ be a first-order algebraic ODE and let $S = \frac{\partial F}{\partial y'}$ be its separant. The singular solutions of $F(x, y, y') = 0$ are defined by the system

$$\{F(x, y, y') = 0, \quad S(x, y, y') = 0\}.$$

‖The visualization is done by `surfex`, which is included as a SINGULAR library, see Labs (2001).

Assume that $y = f(x)$ is a rational singular solution of $F(x, y, y') = 0$. Then $y = f(x)$ must be a rational solution of the first-order autonomous ODE defined by $res_x(F, S) = 0$. This must define a rational curve and we can compute its rational solutions by parametrization. The rational solutions of $res_x(F, S) = 0$ are the candidates for rational singular solutions of $F(x, y, y') = 0$.

E.g. in the above example, $res_x(F, S) = 8y(-y'^3 + 4y^2)$. Solving this differential equation we obtain $y = 0$ and a rational general solution $y = \frac{4}{27}(x + c)^3$, where c is an arbitrary constant. Now, it is clear that $y = 0$ is a singular solution of $F(x, y, y') = 0$. In order that $y = \frac{4}{27}(x + c)^3$ is a singular solution of $F(x, y, y') = 0$, we must have $c = 0$. Hence, $y = \frac{4}{27}x^3$.

2.2.4 Specialize to first-order autonomous algebraic ODEs

We consider autonomous algebraic ODEs of order 1 $F(y, y') = 0$ as a special case of a possibly non-autonomous algebraic ODE. In this section, we show that if $F(y, z) = 0$ is a rational curve, then the associated system of $F(y, y') = 0$ is really simple.

By Feng and Gao (2004, 2006), in order that $F(y, y') = 0$ has a non-trivial rational solution, the algebraic curve $F(y, z) = 0$ must be rational. Suppose that $(f(t), g(t))$ is a proper rational parametrization of the curve $F(y, z) = 0$. Then we immediately have $\mathcal{P}(s, t) = (s, f(t), g(t))$ as a proper parametrization of the solution surface $F(x, y, z) = 0$. This is a special case of a pencil of rational curves, namely, a cylindrical surface. With respect to $\mathcal{P}(s, t)$ the associated system is

$$\left\{ s' = 1, \ \ t' = \frac{g(t)}{f'(t)} \right\}.$$

The second equation of the associated system is again autonomous but of degree 1 in the derivative. Therefore, its rational solution must be either a constant or a linear rational function of the form $\frac{ax + b}{cx + d}$, where a, b, c and d are constants such that $ad - bc \neq 0$.

Let $s = x + C, \ t = t(x)$ be a rational general solution of the above associated system, where C is an arbitrary constant. Then, by algorithm `GENERALSOLVER`, we obtain $y = f(t(x - C))$ as a rational general solution of $F(y, y') = 0$. This means that if we specialize the algorithm `GENERALSOLVER` to first-order autonomous algebraic ODEs, we obtain Algorithm 1 in Feng and Gao (2004). Moreover, in this specialization we can geometrically interpret the reason why we can get a rational general solution from a non-trivial rational solution $y = f(x)$ of $F(y, y') = 0$ by simply taking $y = f(x + C)$ for an arbitrary constant C, which is stated in Theorem 5 in Feng and Gao (2004).

2.2.5 Independence of the proper parametrization

We know that proper rational parametrizations of a rational surface are not unique. Indeed, two parametrizations are different by a birational map of the plane. Let

$$\phi(s_1, t_1) = (\phi_1(s_1, t_1), \phi_2(s_1, t_1))$$

be a birational map of the plane and $\psi(s_2, t_2) = \phi^{-1}(s_2, t_2)$. If $\mathcal{P}(s_1, t_1)$ is a proper parametrization of $F(x, y, z) = 0$, then $(\mathcal{P} \circ \psi)(s_2, t_2)$ is a new proper rational parametrization of $F(x, y, z) = 0$.

Q 6. *How are the associated systems of $F(x, y, y') = 0$ w.r.t. $\mathcal{P}(s_1, t_1)$ and $(\mathcal{P} \circ \psi)(s_2, t_2)$ related to each other?*

Suppose that

$$\begin{cases} s_1' = R_1(s_1, t_1), \\ t_1' = R_2(s_1, t_1), \end{cases}$$

is the associated system of $F(x, y, y') = 0$ w.r.t. $\mathcal{P}(s_1, t_1)$. Then the associated system of $F(x, y, y') = 0$ w.r.t. the proper parametrization $(\mathcal{P} \circ \psi)(s_2, t_2)$ is

$$\begin{pmatrix} s_2' \\ t_2' \end{pmatrix} = J_\phi \cdot \begin{pmatrix} s_1' \\ t_1' \end{pmatrix} = J_\phi \cdot \begin{pmatrix} R_1(s_1, t_1) \\ R_2(s_1, t_1) \end{pmatrix} \Bigg|_{(s_1, t_1) = (\psi_1(s_2, t_2),\ \psi_2(s_2, t_2))} \tag{2.22}$$

where $J_\phi = \begin{pmatrix} \phi_{1s_1} & \phi_{1t_1} \\ \phi_{2s_1} & \phi_{2t_1} \end{pmatrix}$ is the Jacobian matrix of the map ϕ. The two associated systems have the same rational solvability although they are different and the complexity of these systems are not the same.

It can be proven that every birational map of the line is of the form $\phi(x) = \dfrac{ax + b}{cx + d}$, where $a, b, c, d \in \mathbb{K}$ and $ad - bc \neq 0$. Unfortunately, it is not known what are the forms of a birational map of the plane. Therefore, any description on the birational maps of the plane could help us to simplify the associated system and perhaps find the simplest one.

Example 2.2.2. Consider the differential equation

$$F(x, y, y') \equiv y'^2 + 3y' - 2y - 3x = 0.$$

The solution surface $z^2 + 3z - 2y - 3x = 0$ can be parametrized by

$$\mathcal{P}(s, t) = \left(\frac{t}{s} + \frac{2s + t^2}{s^2}, -\frac{1}{s} - \frac{2s + t^2}{s^2}, \frac{t}{s} \right)$$

and also by

$$\mathcal{Q}(s,t) = \left(s, \frac{t^2}{2} - \frac{3}{2}s - \frac{9}{8}, t - \frac{3}{2}\right).$$

The transformation between them is $\phi(s,t) = \left(\frac{t}{s} + \frac{2s+t^2}{s^2}, \frac{t}{s} + \frac{3}{2}\right)$, i.e., $\mathcal{Q} \circ \phi = \mathcal{P}$, with the inverse

$$\phi^{-1} = \psi(s,t) = \left(\frac{8}{-3+8t+4s-4t^2}, \frac{4(2t-3)}{-3+8t+4s-4t^2}\right).$$

Now, the associated system of $F(x,y,y')$ w.r.t. $\mathcal{P}(s,t)$ is $\{s' = st,\ \ t' = s + t^2\}$ while the associated system w.r.t. $\mathcal{Q}(s,t)$ is $\{s' = 1,\ \ t' = 1\}$.

Remark 2.2.2. The properness of a parametrization of the solution surface $F(x,y,z) = 0$ is important. We might consider non-proper parametrizations of $F(x,y,z) = 0$ as well. However, we then have no control on the associated system, i.e., a non-rational solution of the associated system might be mapped into a rational solution of $F(x,y,y') = 0$. For instance, let us consider the differential equation $F \equiv y'^2 - 4y = 0$. The solution surface $z^2 - 4y = 0$ can be parametrized by the improper map $\mathcal{P}(s,t) = \left(s, \frac{t^4}{4}, t^2\right)$. Its associated system is

$$\left\{s' = 1,\ \ t' = \frac{1}{t}\right\}. \tag{2.23}$$

Although this system has a non-rational general solution $(s(x), t(x)) = (x + c, \sqrt{2x})$, its image by $\mathcal{P}(s,t)$ gives us the rational general solution of $y'^2 - 4y = 0$, namely, $y = (x-c)^2$, where c is an arbitrary constant.

Nonetheless, this may well be a way to study more general classes of solutions of algebraic ODEs by parametrization.

2.2.6 A degree bound for rational solutions of the associated system

We have studied the algebraic ODE of order 1, $F(x,y,y') = 0$, provided a proper rational parametrization $\mathcal{P}(s,t) = (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t))$ of the solution surface $F(x,y,z) = 0$. We know that every rational solution $(s(x), t(x))$ of the associated system of $F(x,y,y') = 0$ w.r.t. $\mathcal{P}(s,t)$ satisfies the condition $\chi_1(s(x), t(x)) = x + c$ for some constant c.

Q 7. *What is the relation between* $\deg s(x)$ *and* $\deg t(x)$ *if* $\chi_1(s(x), t(x)) = x$*?*

From the condition $\chi_1(s(x), t(x)) = x$ we can deduce that the degree of $t(x)$ is bounded in terms of the degree of $s(x)$ and the degree of $\chi_1(s,t)$ with respect to s.

Theorem 2.2.9. *Let*

$$\chi_1(s,t) = \frac{a_n(t)s^n + a_{n-1}(t)s^{n-1} + \cdots + a_0(t)}{b_m(t)s^m + b_{m-1}(t)s^{m-1} + \cdots + b_0(t)} \in \mathbb{K}(s,t)$$

be such that $\chi_1(s,t) \notin \mathbb{K}(s)$ *and* $\chi_1(s,t) \notin \mathbb{K}(t)$*, where* $m, n \in \mathbb{N}$ *and* $b_m(t) \neq 0$*. Suppose that* $s(x)$ *and* $t(x)$ *are rational functions in* $\mathbb{K}(x)$ *such that* $\chi_1(s(x), t(x)) = x$*. Let* $\delta = \deg s(x)$*. Then*

$$\deg t(x) \leq 1 + \delta \max\{m, n\}.$$

Proof. We have

$$\chi_1(s(x), t(x)) = x \iff \frac{a_n(t(x))s(x)^n + a_{n-1}(t(x))s(x)^{n-1} + \cdots + a_0(t(x))}{b_m(t(x))s(x)^m + b_{m-1}(t)s(x)^{m-1} + \cdots + b_0(t(x))} = x.$$

We know that for any rational function $t \in \mathbb{K}(x)$, x is algebraic over $\mathbb{K}(t)$ and

$$\deg t(x) = [\mathbb{K}(x) : \mathbb{K}(t)].$$

Therefore, in order to find a degree bound for t, it is enough to find an algebraic equation for x over $\mathbb{K}(t)$. Let $s(x) = \dfrac{P}{Q}$, where $P, Q \in \mathbb{K}[x]$, $Q \neq 0$. Let

$$\delta = \deg s(x) = \max\{\deg P, \deg Q\}, \quad l = \deg Q.$$

We have

$$\begin{aligned} x &= \frac{Q^m}{Q^n} \cdot \frac{(a_n(t)P^n + \cdots + a_0(t)Q^n)}{(b_m(t)P^m + \cdots + b_0(t)Q^m)} \\ &= Q^{m-n} \cdot \frac{(a_n(t)P^n + \cdots + a_0(t)Q^n)}{(b_m(t)P^m + \cdots + b_0(t)Q^m)}. \end{aligned}$$

This equation derives a non-zero algebraic equation of x over $\mathbb{K}(t)$ because $\chi_1(s,t) \notin \mathbb{K}(s)$ and $\chi_1(s,t) \notin \mathbb{K}(t)$. We can compute the degree of x in the above equation regarding $l \leq \delta$.

If $n \geq m$, then

$$\deg t(x) \leq \max\{1 + m\delta + l(n-m), n\delta\} \leq 1 + n\delta.$$

If $n < m$, then

$$\deg t(x) \leq \max\{1 + m\delta, n\delta + l(m-n)\} \leq 1 + m\delta.$$

Therefore, $\deg t(x) \leq 1 + \delta \max\{m, n\}$. □

Of course, the degree of $s(x)$ can also be bounded in the same way by the degree of $t(x)$ and the degree of the first component of $\mathcal{P}(s,t)$ with respect to t.

2.3 A criterion for the existence of a rational general solution

In this section, we derive a criterion for the existence of a rational general solution of the associated system of the equation $F(x, y, y') = 0$. The following lemma can be found in Feng and Gao (2006).

Lemma 2.3.1. *Let $n, m \in \mathbb{N}$. There exists a differential polynomial $D_{n,m}(y)$ such that every rational function*

$$y = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}$$

is a solution of $D_{n,m}(y)$, where a_i, b_j are constants in $\mathbb{K}$. Moreover, the differential polynomial $D_{n,m}(y)$ has only rational solutions.

Definition 2.3.1. The differential polynomial in Lemma 2.3.1 is given by

$$D_{n,m}(y) = \begin{vmatrix} \binom{n+1}{0}y^{(n+1)} & \binom{n+1}{1}y^{(n)} & \cdots & \binom{n+1}{m}y^{(n+1-m)} \\ \binom{n+2}{0}y^{(n+2)} & \binom{n+2}{1}y^{(n+1)} & \cdots & \binom{n+2}{m}y^{(n+2-m)} \\ \vdots & \vdots & \cdots & \vdots \\ \binom{n+1+m}{0}y^{(n+1+m)} & \binom{n+1+m}{1}y^{(n+m)} & \cdots & \binom{n+1+m}{m}y^{(n+1)} \end{vmatrix}.$$

We call $D_{n,m}(y)$ a *Feng-Gao's differential polynomial.*

Using Feng-Gao's differential polynomials we have the following criterion.

Theorem 2.3.2. *Let $M_1, N_1, M_2, N_2 \in \mathbb{K}[s, t]$, $N_1, N_2 \neq 0$. The autonomous system (2.16), i.e.,*

$$\left\{ s' = \frac{M_1(s,t)}{N_1(s,t)}, \quad t' = \frac{M_2(s,t)}{N_2(s,t)} \right\}$$

has a rational general solution $(s(x), t(x))$ with $\deg s(x) \leq n$ and $\deg t(x) \leq m$ if and only if

$$\begin{cases} \text{prem}(D_{n,n}(s), N_1 s' - M_1, N_2 t' - M_2) = 0, \\ \text{prem}(D_{m,m}(t), N_1 s' - M_1, N_2 t' - M_2) = 0. \end{cases} \tag{2.24}$$

Proof. Suppose that the system (2.16) has a rational general solution $(s(x), t(x))$ with $\deg s(x) \leq n$ and $\deg t(x) \leq m$. Then $(s(x), t(x))$ is a solution of both $D_{n,n}(s)$ and $D_{m,m}(t)$. By definition of rational general solutions of the system (2.16) we have

$$\begin{cases} \text{prem}(D_{n,n}(s), N_1 s' - M_1, N_2 t' - M_2) = 0, \\ \text{prem}(D_{m,m}(t), N_1 s' - M_1, N_2 t' - M_2) = 0. \end{cases}$$

Conversely, if these two conditions hold, then $D_{n,n}(s)$ and $D_{m,m}(t)$ belong to

$$\mathcal{I} := \{G \in \mathbb{K}(x)\{s,t\} \mid \operatorname{prem}(G, M_1 s' - N_1, M_2 t' - N_2) = 0\}.$$

Since $\mathcal{I}$ is a prime differential ideal, $\mathcal{I}$ has a generic zero. This generic zero is a zero of $D_{n,n}(s)$ and $D_{m,m}(t)$. By Lemma 2.3.1, these two differential polynomials have only rational solutions. Therefore, the generic zero of I must be rational. □

Remark 2.3.1. If we know a degree bound of the rational solutions of the system (2.16), then Theorem 2.3.2 gives us a criterion for the existence of a rational general solution of the system (2.16). In fact, there is a generic degree bound presented in the next chapter when we study the "invariant algebraic curves" of this associated system.

2.3.1 Application—Linear systems with rational general solutions

We have seen that the associated system of the algebraic ODE $F(x, y, y') = 0$ is an autonomous system. In this section, we consider the linear system of autonomous ODEs of the form

$$\begin{cases} s' = as + bt + e \\ t' = cs + dt + h \end{cases} \tag{2.25}$$

where a, b, c, d, e, h are constants in $\mathbb{K}$.

> **Q 8.** *When does the system has a rational general solution? What are the possible degrees of the rational general solutions?*

In fact, we prove that the rational solutions of that system are polynomials; moreover, their degrees are at most 2. Before studying rational solutions of the system (2.25) we need to introduce the notation of order of an irreducible polynomial in a rational function.

Definition 2.3.2. Let $\mathbb{K}$ be a field. Let $s \in \mathbb{K}(x)$ be a rational function in x. Suppose that s has a complete decomposition as follows

$$s = \frac{A}{p_1^{\alpha_1} \cdots p_n^{\alpha_n}}$$

where $A \in \mathbb{K}[x]$ and p_i are distinct irreducible polynomials over $\overline{\mathbb{K}}$ and $\gcd(A, p_i) = 1$ for all $i = 1, \ldots, n$. The power α_i in this representation of s is called the *order* of s with respect to p_i, denoted by $\operatorname{ord}_{p_i}(s)$. By convention, if an irreducible polynomial p does not effectively appear in the denominator of s, then we define $\operatorname{ord}_p(s) = 0$.

Lemma 2.3.3. *Every rational solution of the linear system* (2.25) *is a polynomial solution.*

Proof. Suppose that $(s(x), t(x))$ is a rational solution of the linear system (2.25). If $s(x)$ or $t(x)$ is not a polynomial, then we can assume without loss of generality that

$$s = \frac{A}{p_1^{\alpha_1} \cdots p_n^{\alpha_n}},$$

where $A \in \mathbb{K}[x]$, p_i are irreducible polynomials over $\overline{\mathbb{K}}$, $\gcd(A, p_i) = 1$ and $\alpha_i > 0$ for all $i = 1, \ldots, n$. Let

$$\beta_i = \operatorname{ord}_{p_i}(t) \geq 0 \quad \forall i = 1, \ldots, n.$$

Since $\alpha_i > 0$, computing the derivative of $s(x)$ we have

$$\operatorname{ord}_{p_i}(s') = \alpha_i + 1 \quad \forall i = 1, \ldots, n.$$

On the other hand,

$$\operatorname{ord}_{p_i}(as + bt + e) \leq \max\{\alpha_i, \beta_i\}, \quad \operatorname{ord}_{p_i}(cs + dt + h) \leq \max\{\alpha_i, \beta_i\}.$$

Let us compare the orders with respect to p_i of the left and the right hand sides of the linear system (2.25). There are two cases as follows.

- Either $\alpha_i \geq \beta_i$, then $\operatorname{ord}_{p_i}(as + bt + e) \leq \alpha_i < \operatorname{ord}_{p_i}(s')$, which is impossible;
- or $0 < \alpha_i < \beta_i$, then $\operatorname{ord}_{p_i}(cs + dt + h) \leq \beta_i < \operatorname{ord}_{p_i}(t')$, which is also impossible.

Therefore, $\alpha_i = 0$ for all $i = 1, \ldots, n$. Thus s is a polynomial. Replacing the role of s and t we also prove that t is a polynomial. Therefore, $(s(x), t(x))$ is a polynomial solution. □

Theorem 2.3.4. *Every rational general solution of the linear system* (2.25) *is a couple of polynomials of degree at most 2.*

Proof. By Lemma 2.3.3, every rational solution of the linear system (2.25) is a polynomial solution. In this case the Gao's differential polynomials for checking polynomial general solutions of the system are of simple forms $s^{(n+1)}$ and $t^{(n+1)}$ for some n. We can write the linear system in the matrix form

$$\begin{pmatrix} s' \\ t' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} e \\ h \end{pmatrix}.$$

Hence

$$\begin{pmatrix} s'' \\ t'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e \\ h \end{pmatrix},$$

$$\vdots$$

$$\begin{pmatrix} s^{(n+1)} \\ t^{(n+1)} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n+1} \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n} \begin{pmatrix} e \\ h \end{pmatrix},$$

for $n \in \mathbb{N}$. By Theorem 2.3.2, the system (2.25) has a polynomial general solution of degree at most n if and only if

$$\begin{cases} \operatorname{prem}(s^{(n+1)}, s' - as - bt - e, t' - cs - dt - h) = 0 \\ \operatorname{prem}(t^{(n+1)}, s' - as - bt - e, t' - cs - dt - h) = 0. \end{cases}$$

or equivalently when

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n+1} = 0 \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n} \begin{pmatrix} e \\ h \end{pmatrix} = 0.$$

We will prove that these relations hold for $n \geq 2$ if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{2} = 0.$$

Then the conclusion of the theorem follows immediately.

Assume that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{2} = 0$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n+1} = 0 \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n} \begin{pmatrix} e \\ h \end{pmatrix} = 0$$

for all $n \geq 2$.

Conversely, let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n} = 0$$

for some $n \geq 1$. Then $ad - bc = 0$ and the Jordan canonical form of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is either $\begin{pmatrix} 0 & 0 \\ 0 & a+d \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In the first case, since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{n} = 0$, we have $a + d = 0$. Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In the second case, we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{2} = 0.$ □

The above proof also tell us the necessary and sufficient conditions of the linear system for having rational general solutions. Namely,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = 0.$$

We can easily find all possibilities of the coefficients a, b, c, d. In fact,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = 0 \iff \begin{cases} a^2 + bc = 0 \\ b(a+d) = 0 \\ c(a+d) = 0 \\ d^2 + bc = 0. \end{cases}$$

Solving this algebraic system we obtain the following cases

- if $b = 0$, then $a = d = 0$;
- if $b \neq 0$, then $a = -d$ and $c = -\dfrac{d^2}{b}$.

Thus the explicit polynomial solutions of the linear system are given by the following table, where C_1, C_2 are arbitrary constants. Note that the last line of the table also covers the

System	Rational general solution
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{cases} s(x) = ex + C_1 \\ t(x) = hx + C_2 \end{cases}$
$\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$	$\begin{cases} s(x) = ex + C_1 \\ t(x) = ce\dfrac{x^2}{2} + (cC_1 + h)x + C_2 \end{cases}$
$\begin{pmatrix} -d & b \\ -\dfrac{d^2}{b} & d \end{pmatrix}$	$\begin{cases} s(x) = \dfrac{hb - ed}{2}x^2 + (bC_1 + e)x + C_2 \\ t(x) = \dfrac{(hb - ed)d}{2b}x^2 + (dC_1 + h)x + \dfrac{d}{b}C_2 + C_1 \end{cases}$

Table 2.1: Linear systems with rational general solutions

other symmetric cases, for instance

$$d = 0 \longmapsto \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}; \quad d = -a,\ b = -\frac{a^2}{c} \longmapsto \begin{pmatrix} a & -\dfrac{a^2}{c} \\ c & -a \end{pmatrix}.$$

We can prove that the solutions in the table are rational general solutions of the corresponding system. For instance, consider a simple system $\{s' = e,\ \ t' = h\}$, where e and h

are constants but not all zero. It turns out that the system has a solution given by

$$s(x) = ex + C_1,\ t(x) = hx + C_2,$$

where C_1, C_2 are arbitrary constants. The implicit defining polynomial of $(s(x), t(x))$ is

$$H(s,t) = hs - et - hC_1 + eC_2.$$

Since the coefficients of $H(s,t)$ contain an arbitrary constant, namely $-hC_1 + eC_2$, it follows from Lemma 2.2.5 that $(s(x), t(x))$ is a rational general solution.

Using a similar argument for the other systems in the table we prove that those solutions are rational general solutions of the corresponding systems.

Chapter 3

Planar rational systems of autonomous ODEs

3.1 Introduction to planar rational systems

In the previous chapter, we are motivated to studying the rational general solutions of the systems of the form (2.16). From the point of view of differential algebra, the solution set of the system (2.16) can be seen as an algebraic differential manifold (Ritt (1950), II, §1). We have described them by means of prime differential ideals in a differential ring. By studying the structure of such prime differential ideals, we are able to see, in the linear cases, when the system (2.16) has a rational general solution and what they are.

On the other hand, we can also study the rational (general) solutions of the system (2.16) from the point of view of algebraic geometry, i.e., by looking at the usual ideal of all polynomials vanishing on a rational solution of the system. In this direction, a treatment on the algebraic solutions of a polynomial system

$$\begin{cases} s' = P(s,t), \\ t' = Q(s,t), \end{cases} \tag{3.1}$$

has already been studying by Darboux (1878). In his work, G. Darboux has introduced the notion of an invariant algebraic curve, i.e., an algebraic relation between $s(x)$ and $t(x)$ of a solution $(s(x), t(x))$ of the polynomial system (3.1). This notion is essential for the Darboux's theory of integrability of a polynomial system. By Darboux, the system is integrable iff it has a first integral, i.e., a non-constant function such that its values on every solution of the system is constant. Invariant algebraic curves are the main ingredient to build up a first integral of the system.

Recall that the associated system in Chapter 2 is a rational system of autonomous

ODEs of the form (2.16), i.e.,

$$\left\{ s' = \frac{M_1(s,t)}{N_1(s,t)},\ \ t' = \frac{M_2(s,t)}{N_2(s,t)} \right\}.$$

The rational system (2.16) and the polynomial system defined by $P = M_1N_2$ and $Q = M_2N_1$ have the same invariant algebraic curves and the same first integrals. Therefore, the Darboux's theory is applied to study rational (general) solutions of the system (2.16). In fact, we apply the theory of rational parametrization of algebraic curves for the invariant algebraic curves of the system in order to obtain an explicit rational (general) solution of the system.

Goal 1. *The goal of this chapter is to study the existence of a rational general solution of the system* (2.16) *and in the affirmative case we determine an explicit one.*

The fact is that any rational solution of the system (2.16) is corresponding to an invariant algebraic curve of the system. Furthermore, this curve is rational. Therefore, we first need to determine the invariant algebraic curves of the system. Then we apply the theory of rational parametrization of algebraic curves for those invariant algebraic curves. In the end, we give an explicit procedure to computing a rational general solution of the system (2.16). The result of this chapter is based on Ngô and Winkler (2011)

Before going to details, we notice that the polynomial system (3.1) has been discussed in the context of holomorphic singular foliations of the complex projective plane $\mathbb{CP}^2$ (Darboux (1878); Jouanolou (1979); Lins Neto (1988); Carnicer (1994)). Most of the time we stay in the affine plane but in some discussions we need a result—the degree bound of an invariant algebraic curve—that holds in the complex projective plane. Therefore, in this chapter, we also mention the description of the system (3.1) in the complex projective plane.

Note that, one can derive from the system (2.16) and the polynomial system (3.1) to the single differential equation

$$\frac{dt}{ds} = \frac{Q(s,t)}{P(s,t)}, \tag{3.2}$$

or the 1-form

$$Q(s,t)ds - P(s,t)dt = 0, \tag{3.3}$$

or the polynomial vector field

$$\mathcal{D} := P\frac{\partial}{\partial s} + Q\frac{\partial}{\partial t}. \tag{3.4}$$

However, the correspondence is not one-to-one. Hence, from the equation (3.2), we can not construct the system (2.16) or the polynomial system (3.1). But it is enough to have the form (3.2) or (3.3) or (3.4) for studying the invariant algebraic curves of the system (2.16) or of the polynomial system (3.1).

3.2 Invariant algebraic curves of a planar rational system

In this section, we present the notion of invariant algebraic curves of the system (2.16) and a method to determining these curves.

Let $P = M_1N_2$ and $Q = M_2N_1$, be two polynomials in $\mathbb{K}[s,t]$; denote that $\mathcal{D} = P\dfrac{\partial}{\partial s} + Q\dfrac{\partial}{\partial t}$. Suppose that $(s(x), t(x))$ is a solution of the system (2.16) such that there exists an irreducible polynomial $G(s,t)$ with $G(s(x), t(x)) = 0$. Then we have

$$\mathcal{D}G = GK,$$

where K is some polynomial of degree at most $m-1$ and $m = \max\{\deg(P), \deg(Q)\}$.

Definition 3.2.1. An algebraic curve $G(s,t) = 0$ is called an *invariant algebraic curve* of the system (2.16) iff

$$\mathcal{D}G = GK,$$

for some polynomial K. The polynomial K is called the *cofactor* of G.

By definition, the invariant algebraic curves of the rational system (2.16) and that of the polynomial (3.1), where $P = M_1N_2$ and $Q = M_2N_1$, are the same. Therefore, when we are interested in the invariant algebraic curves of those systems, we will consider the analysis on polynomial systems.

Definition 3.2.2. An invariant algebraic curve $G(s,t) = 0$ of the system (2.16) is called a *general invariant algebraic curve* iff $G(s,t)$ is a monic polynomial w.r.t. a lexicographic order of terms in s, t and there exists a coefficient of G such that it is transcendental over $\mathbb{K}$.

In this case, $G(s,t) = 0$ can be seen as either one curve over $\overline{\mathbb{K}(c)}$, where c is a transcendental constant over $\mathbb{K}$, or a family of curves over $\mathbb{K}$. By Lemma 2.2.5, general invariant algebraic curves would be the algebraic curves of a potential general solution of the system (2.16).

Lemma 3.2.1. *Let $G(s,t) = \prod_{i=1}^{l} G_i^{n_i}$ be the decomposition of $G(s,t)$ into relatively prime irreducible factors over $\mathbb{K}$. Then $G(s,t) = 0$ is an invariant algebraic curve of the system (2.16) with cofactor $K(s,t)$ if and only if the curve $G_i(s,t) = 0$ is an invariant algebraic curve of that system with cofactor K_i for all $i = 1, \cdots, l$ and $K = \sum_{i=1}^{l} n_i K_i$.*

Proof. Suppose that $G(s,t)=0$ is an invariant algebraic curve, i.e., we have

$$\mathcal{D}G=\sum_{i=1}^{l} n_i G_i^{n_i-1}\mathcal{D}G_i \prod_{j\neq i} G_j^{n_j} = G\cdot K,$$

where K is the cofactor of G. It implies that G_i devides $\mathcal{D}G_i\prod_{j\neq i}G_j^{n_j}$. Since G_i and G_j are relatively prime for $i\neq j$, we must have that G_i devides $\mathcal{D}G_i$. In other words, $G_i=0$ is an invariant algebraic curve for all $i=1,\ldots,l$.

Conversely, let $G_i=0$ be an invariant algebraic curve with cofactor K_i. Then

$$\begin{aligned}\mathcal{D}G &=\sum_{i=1}^{l} n_i G_i^{n_i-1}\mathcal{D}G_i \prod_{j\neq i} G_j^{n_j}\\ &=\prod_{i=1}^{l} G_i^{n_i}\left(\sum_{i=1}^{l} n_i K_i\right).\end{aligned}$$

Hence, $G=0$ is an invariant algebraic curve with cofactor $K=\sum_{i=1}^{l} n_iK_i$. □

Therefore, from now on, we only consider the irreducible invariant algebraic curves of the system (2.16). Computing an irreducible invariant algebraic curve $G(s,t)=0$ of the system (2.16) can be performed via undetermined coefficients method as long as an upper bound for the degree of the polynomial $G(s,t)$ is setting up.

Let $H=\gcd(P,Q)$, $P=P_1H$ and $Q=Q_1H$. Then every invariant algebraic curve of the system

$$\begin{cases} s'=P_1(s,t),\\ t'=Q_1(s,t),\end{cases} \tag{3.5}$$

is an invariant algebraic curve of the system (2.16). Conversely, suppose that $G(s,t)=0$ is an irreducible invariant algebraic curve of the system (2.16). Then

$$(G_sP_1+G_tQ_1)H=GK$$

for some polynomial K. Since $G(s,t)$ is irreducible, either $G|H$ or $G|(G_sP_1+G_tQ_1)$. In the latter case, $G(s,t)=0$ is an invariant algebraic curve of the system (3.5). In the first case, $G(s,t)$ is an irreducible factor of $H(s,t)$ and for any parametrization $(s(x),t(x))$ of $G(s,t)=0$ we have

$$P(s(x),t(x))=0=Q(s(x),t(x)).$$

In this case, a parametrization $(s(x),t(x))$ of $G(s,t)=0$ is a solution of the system (2.16) only if $s(x)$ and $t(x)$ are constants, i.e., any point of the curve $G(s,t)=0$ gives a trivial rational solution of the system (2.16).

Example 3.2.1. Consider the polynomial differential system

$$\begin{cases} s' = st, \\ t' = s + t^2. \end{cases} \tag{3.6}$$

We first ask for the invariant algebraic curves of degree 1. Consider the graded lexicographic order with $s > t$. Then there are two case distinctions, namely,

$$G(s,t) = t + c, \quad G(s,t) = s + bt + c.$$

The first polynomial can not define an invariant algebraic curve because

$$G_sP + G_tQ = s + t^2$$

is not divisible by G. Now we consider the second polynomial, and the remainder of the division of $G_sP + G_tQ$ by G is

$$(-c - b^2)t - bc.$$

It follows that $G(s,t) = s + bt + c$ defines an invariant algebraic curve if and only if $b = c = 0$. Therefore, $G(s,t) = s$ is an invariant algebraic curve of degree 1.

Similarly, we ask for the invariant algebraic curves of degree 2. Again take the graded lexicographic order with $s > t$. There are three case distinctions, namely,

$$G(s,t) = t^2 + ds + et + f, \quad G(s,t) = st + ct^2 + ds + et + f,$$

$$G(s,t) = s^2 + bst + ct^2 + ds + et + f.$$

If $G(s,t) = t^2 + ds + et + f$, then the remainder of the division of $G_sP + G_tQ$ by G is

$$(2 - d)st + (de + e)s + (e^2 - 2f)t + ef.$$

So we need to have $d = 2$ and $e = f = 0$. Hence, $G(s,t) = t^2 + 2s$ is an invariant algebraic curve of the system. With the same procedure we can see that $G(s,t) = st + ct^2 + ds + et + f$ is not an invariant algebraic curve for any choice of its coefficients; and $G(s,t) = s^2 + bst + ct^2 + ds + et + f$ is an invariant algebraic curve if and only if $b = e = f = 0$ and $d = 2c$, i.e., $G(s,t) = s^2 + ct^2 + 2cs$, where c is an arbitrary constant.

The family of irreducible curves $s^2 + ct^2 + 2cs = 0$ corresponds to the level curves of the surface $z = \frac{s^2}{2s+t^2}$. Later, the function $\frac{s^2}{2s+t^2}$ is a rational first integral of the given polynomial system. Hence, we have two ways of visualizing this family of curves as in Figure 3.1.

Of course, we can keep on increasing the degree of the curve for computing the irre-

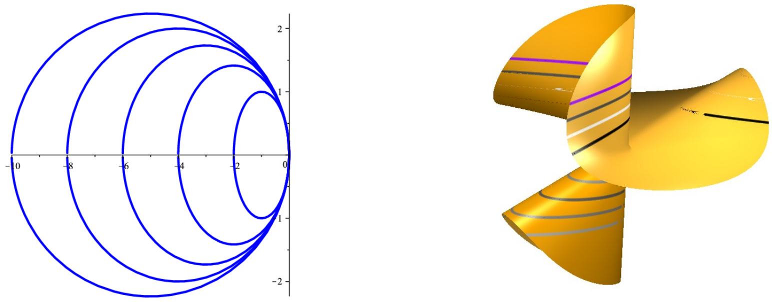

Figure 3.1: Family of irreducible invariant algebraic curves: $s^2 + ct^2 + 2cs = 0$

ducible invariant algebraic curves of this system. In this example, however, we will see that a theorem by Darboux guarantees that the system has no irreducible invariant algebraic curve of degree higher than 2.

The method consists of two steps: computing a normal form of $\mathcal{D}G$ in the ideal generated by G w.r.t. an ordering and then solving the result algebraic system on the coefficients of G. In Man (1993), one can find a discussion on the efficiency of different implementations of computing invariant algebraic curves in some computer algebra systems (MACSYMA and REDUCE).

3.3 Rational solutions of planar rational systems of first-order autonomous ODEs

In this section, we give an algorithm to determine a rational solution of the system (2.16). A *rational solution* of the system (2.16) is a pair of rational functions $\mathcal{C}(x) = (s(x), t(x))$ satisfying the system (2.16).

Definition 3.3.1. A rational solution $(s(x), t(x))$ of (2.16) is said to be *trivial* if both $s(x)$ and $t(x)$ are constants.

A trivial solution of (2.16) can be easily found by intersecting the two algebraic curves $M_1(s,t) = 0$ and $M_2(s,t) = 0$. Otherwise, a non-trivial rational solution $(s(x), t(x))$ of (2.16) defines a rational curve. Let us see what properties of that algebraic curve are.

Lemma 3.3.1. *Let $\mathcal{C}(x) = (s(x), t(x))$ be a non-trivial rational solution of the system* (2.16)*. Let $G(s,t)$ be the defining polynomial of the curve parametrized by $\mathcal{C}(x)$. Then*

$$G_s M_1 N_2 + G_t M_2 N_1 = GK,$$

where G_s and G_t are the partial derivatives of G w.r.t. s and t; K is some polynomial.

Proof. Since $G(s,t)$ is the defining polynomial of the curve parametrized by $(s(x),t(x))$, we have

$$G(s(x),t(x)) = 0.$$

Differentiating this equation with respect to x we obtain

$$G_s(s(x),t(x)) \cdot s'(x) + G_t(s(x),t(x)) \cdot t'(x) = 0.$$

$\mathcal{R}(x) = (s(x),t(x))$ is a solution of the system, so we have

$$G_s(s(x),t(x)) \cdot \frac{M_1(s(x),t(x))}{N_1(s(x),t(x))} + G_t(s(x),t(x)) \cdot \frac{M_2(s(x),t(x))}{N_2(s(x),t(x))} = 0.$$

Hence, the polynomial $G_sM_1N_2+G_tM_2N_1$ is in the ideal of the curve generated by $G(s,t)$. In other words, we have

$$G_sM_1N_2 + G_tM_2N_1 = GK$$

for some polynomial K. □

Every non-trivial rational solution of the system (2.16) determines a rational curve, which is also an invariant algebraic curve of the system. Therefore, we first look for the invariant algebraic curves of the system and then parametrize them to obtain rational solutions. Of course, if none of the invariant algebraic curves is rational, then we immediately conclude that there is no rational solution.

Definition 3.3.2. An invariant algebraic curve $G(s,t) = 0$ of the rational system (2.16) is called a *rational invariant algebraic curve* iff $G(s,t) = 0$ is a rational curve.

Lemma 3.3.2. *Let $G(s,t) = 0$ be an irreducible rational invariant algebraic curve of the system* (2.16)*. Let $(s(x),t(x))$ be a rational parametrization of the curve $G(s,t) = 0$. Then we have*

$$s'(x) \cdot M_2(s(x),t(x))N_1(s(x),t(x)) = t'(x) \cdot M_1(s(x),t(x))N_2(s(x),t(x)).$$

Moreover, if $G \nmid N_1$ and $G \nmid N_2$, then

$$s'(x) \cdot \frac{M_2(s(x),t(x))}{N_2(s(x),t(x))} = t'(x) \cdot \frac{M_1(s(x),t(x))}{N_1(s(x),t(x))}.$$

Proof. Since $G(s(x),t(x)) = 0$, we have

$$G_s(s(x),t(x))s'(x) + G_t(s(x),t(x))t'(x) = 0.$$

Moreover, $G(s,t)=0$ is an invariant algebraic curve, so we have

$$G_s(s(x),t(x))M_1(s(x),t(x))N_2(s(x),t(x))+G_t(s(x),t(x))M_2(s(x),t(x))N_1(s(x),t(x))=0.$$

Note that the irreducibility of G implies $(G_s(s(x),t(x)),G_t(s(x),t(x)))\neq(0,0)$. Therefore,

$$\begin{vmatrix} s'(x) & t'(x) \\ M_1(s(x),t(x))\cdot N_2(s(x),t(x)) & M_2(s(x),t(x))\cdot N_1(s(x),t(x)) \end{vmatrix}=0.$$

Moreover, if $G\nmid N_1$ and $G\nmid N_2$, then $N_1(s(x),t(x))\neq 0$ and $N_2(s(x),t(x))\neq 0$. Hence,

$$s'(x)\cdot\frac{M_2(s(x),t(x))}{N_2(s(x),t(x))}=t'(x)\cdot\frac{M_1(s(x),t(x))}{N_1(s(x),t(x))}.$$

□

The lemma tells us that not every rational parametrization of a rational invariant algebraic curve will provide a rational solution of the system. But they are good candidates for rational solutions of the system. Therefore, we still have to determine whether any of the infinitely many rational parametrizations leads to a solution of the system. We know that a rational parametrization of a curve is completely determined by reparametrization of a proper parametrization of the curve (see Appendix A).

Q 9. *Which reparametrizations of a proper rational parametrization of an invariant algebraic curve lead to solutions of the system* (2.16)*?*

Definition 3.3.3. A rational invariant algebraic curve of the system (2.16) is called a *rational solution curve* iff it possesses a rational parametrization which is a solution of the system.

From now on, we are only interested in non-trivial rational solutions of the system (2.16). Let us recall that if $(s(x),t(x))$ is a rational parametrization of an algebraic curve $G(s,t)=0$, then at least one of the components of $(s(x),t(x))$ must be non-constant.

The following theorem provides a necessary and sufficient condition for a rational invariant algebraic curve to be a rational solution curve.

Theorem 3.3.3. *Let $G(s,t)=0$ be a rational invariant algebraic curve of the system* (2.16) *such that $G\nmid N_1$ and $G\nmid N_2$. Let $(s(x),t(x))$ be a proper rational parametrization of $G(s,t)=0$. Then $G(s,t)=0$ is a rational solution curve of the system* (2.16) *if and only if one of the following differential equations has a rational solution $T(x)$:*

1. $T'=\dfrac{1}{s'(T)}\cdot\dfrac{M_1(s(T),t(T))}{N_1(s(T),t(T))}$ *when* $s'(x)\neq 0$,
2. $T'=\dfrac{1}{t'(T)}\cdot\dfrac{M_2(s(T),t(T))}{N_2(s(T),t(T))}$ *when* $t'(x)\neq 0$.

If there is such a rational solution $T(x)$, then the rational solution of the system (2.16) *corresponding to $G(s,t)=0$ is given by $(s(T(x)),\ t(T(x)))$.*

Proof. Assume that $(\bar{s}(x),\bar{t}(x))$ is a rational solution of the system (2.16) corresponding to $G(s,t)=0$, i.e.,

$$\begin{cases} \bar{s}'(x) = \dfrac{M_1(\bar{s}(x),\bar{t}(x))}{N_1(\bar{s}(x),\bar{t}(x))}, \\ \bar{t}'(x) = \dfrac{M_2(\bar{s}(x),\bar{t}(x))}{N_2(\bar{s}(x),\bar{t}(x))}. \end{cases}$$

Since $(s(x),t(x))$ is a proper parametrization of $G(s,t)=0$, there exists a rational function $T(x)$ such that

$$\bar{s}(x) = s(T(x)),\ \ \bar{t}(x) = t(T(x)).$$

It implies that

$$\begin{cases} \bar{s}'(x) = s'(T(x)) \cdot T'(x), \\ \bar{t}'(x) = t'(T(x)) \cdot T'(x). \end{cases}$$

Therefore,

$$T'(x) \cdot s'(T(x)) = \frac{M_1(\bar{s}(x),\bar{t}(x))}{N_1(\bar{s}(x),\bar{t}(x))}$$

and

$$T'(x) \cdot t'(T(x)) = \frac{M_2(\bar{s}(x),\bar{t}(x))}{N_2(\bar{s}(x),\bar{t}(x))}.$$

When $s(x)$ or $t(x)$ are non-constants, we have

$$T'(x) = \frac{1}{s'(T(x))} \cdot \frac{M_1(\bar{s}(x),\bar{t}(x))}{N_1(\bar{s}(x),\bar{t}(x))} \ \text{ or } \ T'(x) = \frac{1}{t'(T(x))} \cdot \frac{M_2(\bar{s}(x),\bar{t}(x))}{N_2(\bar{s}(x),\bar{t}(x))},$$

respectively. Conversely, assume w.l.o.g. that $s(x)$ is non-constant and $T(x)$ is a rational solution of the first differential equation. By Lemma 3.3.2 we have

$$s'(T) \cdot \frac{M_2(s(T),t(T))}{N_2(s(T),t(T))} = t'(T) \cdot \frac{M_1(s(T),t(T))}{N_1(s(T),t(T))}.$$

If $t'(T)=0$, then $\dfrac{M_2(s(T),t(T))}{N_2(s(T),t(T))} = 0$ and $t(x)=c$, for some constant c. It is obvious that $(s(T(x)),c)$ is a rational solution of the system (2.16). Hence $G(s,t)=0$ is a rational solution curve. If $t'(T)\neq 0$, then

$$\frac{1}{t'(T)} \cdot \frac{M_2(s(T),t(T))}{N_2(s(T),t(T))} = \frac{1}{s'(T)} \cdot \frac{M_1(s(T),t(T))}{N_1(s(T),t(T))}.$$

Therefore, $T(x)$ is also a rational solution of the second differential equation. It follows that $(s(T(x)),t(T(x)))$ is a rational solution of the system (2.16). Hence $G(s,t)=0$ is a rational solution curve. □

We note that if $s(x)$ and $t(x)$ are both non-constant, the two differential equations in Theorem 3.3.3 are the same because the expressions on the right hand side are equal by Lemma 3.3.2. According to Theorem 3.3.3, assuming that we are in case (1), we need to compute a rational solution of the autonomous differential equation of order 1 and of degree 1 in T'

$$T'(x) = \frac{1}{s'(T)} \cdot \frac{M_1(s(T), t(T))}{N_1(s(T), t(T))}.$$

Those features of the differential equation are inherited by the associated system. In what follows, we see that it is simple to deal with the rational solvability of this differential equation.

In the next theorem, we prove that the rational solvability of this differential equation does not depend on the choice of a proper parametrization of the rational invariant algebraic curve $G(s,t) = 0$.

Theorem 3.3.4. *Let $G(s,t)$ be a rational invariant algebraic curve of the system* (2.16) *such that $G \nmid N_1$ and $G \nmid N_2$. Let $\mathcal{P}_1(x) = (s_1(x), t_1(x))$ and $\mathcal{P}_2(x) = (s_2(x), t_2(x))$ be two proper rational parametrizations of the curve $G(s,t) = 0$ such that $s_1'(x) \neq 0$ and $s_2'(x) \neq 0$. Then the two autonomous differential equations*

$$T_1' = \frac{1}{s_1'(T_1)} \cdot \frac{M_1(s_1(T_1), t_1(T_1))}{N_1(s_1(T_1), t_1(T_1))} \tag{3.7}$$

and

$$T_2' = \frac{1}{s_2'(T_2)} \cdot \frac{M_1(s_2(T_2), t_2(T_2))}{N_2(s_2(T_2), t_2(T_2))} \tag{3.8}$$

have the same rational solvability, i.e., one of them has a rational solution if and only if the other one has. Moreover, we can choose T_1 and T_2 such that

$$\mathcal{P}_1(T_1) = \mathcal{P}_2(T_2).$$

Proof. Suppose that (3.7) has a rational solution $T_1(x)$. Then the rational solution of (2.16) corresponding to $G(s,t) = 0$ is $(s_1(T_1(x)), t_1(T_1(x)))$. Since $(s_2(x), t_2(x))$ is a proper rational parametrization of the same curve $G(s,t) = 0$, there exists a rational function $T_2(x)$ such that

$$s_2(T_2(x)) = s_1(T_1(x)), \;\; t_2(T_2(x)) = t_1(T_1(x)).$$

Hence,

$$s_2'(T_2(x))T_2'(x) = s_1'(T_1(x))T_1'(x) = \frac{M_1(s_1(T_1), t_1(T_1))}{N_1(s_1(T_1), t_1(T_1))} = \frac{M_1(s_2(T_2), t_2(T_2))}{N_1(s_2(T_2), t_2(T_2))}.$$

This means that $T_2(x)$ is a rational solution of (3.8). □

Theorem 3.3.5. *Suppose that* $s(x)$ *is a non-constant rational function and* $N_1(s(x), t(x)) \neq 0$. *Then every rational solution of*

$$T' = \frac{1}{s'(T)} \cdot \frac{M_1(s(T), t(T))}{N_1(s(T), t(T))}$$

is of the form $T(x) = \dfrac{ax+b}{cx+d}$, *where* a, b, c *and* d *are constants. In particular, every non-trivial rational solution of the system* (2.16) *is proper in the sense of proper parametrization.*

Proof. Assume that $T(x)$ is a non-constant rational solution of the above differential equation. Then, by Theorem 3.7 in Feng and Gao (2006), $(T(x), T'(x))$ forms a proper parametrization of the algebraic curve $H(T, U) = 0$ defined by the numerator of

$$s'(T) \cdot N_1(s(T), t(T)) \cdot U - M_1(s(T), t(T)).$$

Using the degree bound of proper parametrizations in Sendra and Winkler (2001) and observing that the degree of $H(T, U)$ with respect to U is 1, we see that the degree of $T(x)$ must be 1. Therefore, by Theorem 3.3.3, a non-trivial rational solution of the system (2.16) is a composition of a proper parametrization with a non-constant linear rational function, hence a proper parametrization. □

Remark 3.3.1. There is another way of seeing this result. By Theorem 3.7 in Feng and Gao (2006), if $T(x)$ is a non-constant rational function, then we have $\mathbb{K}(x) = \mathbb{K}(T(x), T'(x))$. If $T'(x) = f(T(x))$, where $f(T)$ is a non-zero rational function in $\mathbb{K}(T)$, then

$$\mathbb{K}(x) = \mathbb{K}(T(x), T'(x)) = \mathbb{K}(T(x), f(T(x))) = \mathbb{K}(T(x)).$$

Therefore, $T(x)$ is a linear rational function in x.

By this theorem we can always decide whether the differential equation for $T(x)$ has a rational solution. Therefore, we can decide whether an invariant algebraic curve $G(s, t) = 0$ is a rational solution curve.

Note that if $(s(x), t(x))$ is a rational solution of the system (2.16), then, because of the autonomy of (2.16),

$$(s(x + \tilde{c}), t(x + \tilde{c}))$$

is also a rational solution of the system. In fact, this is the only way to generate rational solutions from the same rational solution curve.

Theorem 3.3.6. *Let $(s_1(x), t_1(x))$ and $(s_2(x), t_2(x))$ be non-trivial rational solutions of the differential system* (2.16) *corresponding to the same rational invariant algebraic curve. Then there exists a constant $\tilde{c}$ such that*

$$(s_1(x+\tilde{c}), t_1(x+\tilde{c})) = (s_2(x), t_2(x)).$$

Proof. As a corollary of Theorem 3.3.3 and Theorem 3.3.5, we have proven that these solutions are proper. Since $(s_1(x), t_1(x))$ and $(s_2(x), t_2(x))$ are rational parametrizations of the same invariant algebraic curve, there exists a linear rational function $T(x)$ such that

$$(s_2(x), t_2(x)) = (s_1(T(x)), t_1(T(x))).$$

Hence

$$\begin{cases} s_1'(T(x))T'(x) = s_2'(x) = \dfrac{M_1(s_2(x), t_2(x))}{N_1(s_2(x), t_2(x))} = \dfrac{M_1(s_1(T(x)), t_1(T(x)))}{N_1(s_1(T(x)), t_1(T(x)))} = s_1'(T(x)) \\ t_1'(T(x))T'(x) = t_2'(x) = \dfrac{M_2(s_2(x), t_2(x))}{N_2(s_2(x), t_2(x))} = \dfrac{M_2(s_1(T(x)), t_1(T(x)))}{N_2(s_1(T(x)), t_1(T(x)))} = t_1'(T(x)). \end{cases} \tag{3.9}$$

It follows that $T'(x) = 1$. Therefore, $T(x) = x + \tilde{c}$ for some constant $\tilde{c}$. In fact, we can compute the precise transformation $T(x) = (\mathcal{P}_1^{-1} \circ \mathcal{P}_2)(x)$, where $\mathcal{P}_1(x) = (s_1(x), t_1(x))$ and $\mathcal{P}_2(x) = (s_2(x), t_2(x))$. □

Remark 3.3.2. Let $(s(x), t(x))$ be a rational solution of the autonomous system (2.16). Then for any constant $\tilde{c}$, $(s(x+\tilde{c}), t(x+\tilde{c}))$ is also a rational solution of that system. In a sense, the latter solution is more general because we can evaluate any value for the constant $\tilde{c}$. From the point of view of parametrization, however, these two rational solutions parametrize the same algebraic curve. Later on, when we map this solution curve into a solution curve of the differential equation $F(x, y, y') = 0$, the constant $\tilde{c}$ will be eliminated and it plays no role in generating a rational general solution of $F(x, y, y') = 0$. Therefore, we can simply skip that arbitrary constant, appearing in the shifting way, of a the general solution of (2.16).

3.4 Algorithm and examples

We give our algorithm to compute a rational solution of the system (2.16) from an irreducible invariant algebraic curve of the system. If we apply the algorithm to a general invariant algebraic curve, then, in the affirmative case, its output is a rational general solution.

Algorithm `RATSOLVE`
Input: The system (2.16) and an irreducible invariant algebraic curve $G(s,t)=0$ of the system such that $G \nmid N_1$ and $G \nmid N_2$.
Output: The corresponding rational solution of (2.16), if any.

1. `if` $G(s,t)=0$ is not a rational curve, `then` return "there is no rational solution corresponding to $G(s,t)=0$."
2. `else` compute a proper rational parametrization $(s(x),t(x))$ of $G(s,t)=0$.
3. `if` $s'(x) \not\equiv 0$, then find the rational solution of the autonomous differential equation
$$T' = \frac{1}{s'(T)} \cdot \frac{M_1(s(T),t(T))}{N_1(s(T),t(T))}.$$
4. `else` find the rational solution of the autonomous differential equation
$$T' = \frac{1}{t'(T)} \cdot \frac{M_2(s(T),t(T))}{N_2(s(T),t(T))}.$$
5. `if` there exists $T(x)$ (a linear rational function), `then` return
$$(s(T(x)),t(T(x)))$$
6. `else` return "there is no rational solution corresponding to $G(s,t)=0$."

Example 3.4.1. Consider the rational differential system

$$\begin{cases} s' = \dfrac{-2(-(t-1)^2+s^2)(t-1)^2}{((t-1)^2+s^2)^2}, \\ t' = \dfrac{-4(t-1)^3 s}{((t-1)^2+s^2)^2}. \end{cases} \tag{3.10}$$

First we compute the set of invariant algebraic curves of degree less than or equal to 2,

$$\left\{t-1=0, s+\sqrt{-1}(t-1)=0, s-\sqrt{-1}(t-1)=0, s^2+t^2+(-1-c)t+c=0\right\},$$

where c is an arbitrary constant.

For computing rational solutions of the system we will not consider the two invariant algebraic curves $s+\sqrt{-1}(t-1)=0$ and $s-\sqrt{-1}(t-1)=0$ because they are divisors of the denominators of the system.

The invariant algebraic curve $t-1=0$ can be parametrized by $(x,1)$. The corresponding differential equation for reparametrization is $T'=0$. Hence, $T(x)=c$, where

c is an arbitrary constant. This implies that $s(x) = c, t(x) = 1$ is a rational solution corresponding to the rational solution line $t - 1 = 0$.

It remains to consider the general invariant algebraic curve $s^2 + t^2 + (-1-c)t + c = 0$. This determines a rational curve in $\mathbb{A}^2(\overline{\mathbb{K}(c)})$, having the proper rational parametrization

$$\mathcal{P}(x) = \left(\frac{(c-1)x}{1+x^2}, \frac{cx^2+1}{1+x^2} \right).$$

By the algorithm RATSOLVE, the corresponding autonomous differential equation for reparametrization is $T' = \dfrac{-2T^2}{c-1}$. Hence $T(x) = \dfrac{c-1}{2x}$. Now we subsitute $T(x)$ into $\mathcal{P}(x)$ to obtain a rational (general) solution of the system (3.10), namely,

$$s(x) = \frac{2(c-1)^2 x}{4x^2 + (c-1)^2}, \quad t(x) = \frac{c(c-1)^2 + 4x^2}{4x^2 + (c-1)^2}.$$

Later, we can prove that the system has no irreducible invariant algebraic curve of degree higher than 2 (by Darboux's theorem).

Remark 3.4.1. Proper rational parametrizations of a rational curve are not unique. We can also parametrize the curve $s^2 + t^2 + (-1-c)t + c = 0$ by

$$\mathcal{P}_1(x) = \left(\frac{-ci + (i+ic)x - ix^2}{1+c-2x}, \frac{c-x^2}{1+c-2x} \right),$$

where i is the imaginary unit. In this case, we obtain another rational general solution, namely,

$$s(x) = \frac{(2ix - c + 1)(c-1)^2}{4x(ix - c + 1)}, \quad t(x) = \frac{i(4x^2 + 4i(c-1)x + (c-1)^3)}{4x(ix - c + 1)}.$$

This solution is transformable into the first one by the change of variable

$$\varphi(x) = x + \frac{i(c-1)}{2}.$$

Here we give another example for a complete algorithm combining the two algorithms GENERALSOLVER and RATSOLVE.

Example 3.4.2. Considering the differential equation

$$F(x, y, y') \equiv y'^2 + 3y' - 2y - 3x = 0. \tag{3.11}$$

The corresponding algebraic surface $z^2 + 3z - 2y - 3x = 0$ can be parametrized by

$$\mathcal{P}_0(s,t) = \left(\frac{t}{s} + \frac{2s+t^2}{s^2}, -\frac{1}{s} - \frac{2s+t^2}{s^2}, \frac{t}{s}\right).$$

This is a proper parametrization and the corresponding associated system is

$$\begin{cases} s' = st, \\ t' = s + t^2. \end{cases}$$

We compute the set of irreducible invariant algebraic curves of the system and obtain

$$\{s = 0, t^2 + 2s = 0, s^2 + ct^2 + 2cs = 0 \mid c \text{ is an arbitrary constant}\}.$$

The general invariant algebraic curve $s^2 + ct^2 + 2cs = 0$ can be parametrized by

$$\mathcal{Q}(x) = \left(-\frac{2cx^2}{x^2+c}, -\frac{2cx}{x^2+c}\right).$$

By the algorithm **RATSOLVE**, we have to solve an auxiliary differential equation for the reparametrization, namely:

$$T' = \frac{1}{\mathcal{Q}_1(T)'}\mathcal{Q}_1(T)\mathcal{Q}_2(T) = -T^2.$$

Hence, $T(x) = \frac{1}{x}$. So the rational general solution of the associated system is

$$s(x) = \mathcal{Q}_1(T(x)) = -\frac{2c}{1+cx^2}, \;\; t(x) = \mathcal{Q}_2(T(x)) = -\frac{2cx}{1+cx^2}.$$

We observe that

$$\chi_1(s(x), t(x)) = x - \frac{1}{c}.$$

Therefore, according to algorithm **GENERALSOLVER**, the rational general solution of (3.11) is

$$y = \chi_2\left(s\left(x + \frac{1}{c}\right), t\left(x + \frac{1}{c}\right)\right) = \frac{1}{2}x^2 + \frac{1}{c}x + \frac{1}{2c^2} + \frac{3}{2c},$$

which, after a change of parameter, can be written as

$$y = \frac{1}{2}((x+c)^2 + 3c).$$