Chapter 2

Rational solutions of autonomous first-order AODEs

An *algebraic ordinary differential equation (AODE)* is a polynomial relation between a function $y(x)$, some (finitely many) of its derivatives $y'(x), y''(x), \ldots$ and the variable of differentiation x. So there is a polynomial F (say, over the field $\mathbb{Q}$) such that the AODE looks like

$$F(x, y, y', \ldots, y^{(n)}) = 0 \ .$$

F defines the differential equation; but often, by abuse of notation, F itself is called the AODE. W.l.o.g. we may consider F to be irreducible; other wise factor F and solve the AODEs defined by the factors.
The highest n s.t. $y^{(n)}$ actually appears in F is called the *order* of the AODE F.
An AODE is *autonomous* iff it is independent from x; i.e., it is of the form

$$F(y, y', \ldots, y^{(n)}) = 0 \ .$$

A rational function $y(x) \in \overline{\mathbb{Q}}[x]$ ($\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$, i.e., the field of algebraic numbers) satisfying $F(x, y, y', \ldots, y^{(n)}) = 0$ is called a *rational solution* of the AODE F.

Let I be an ideal in $\mathbb{Q}(x)[y, y', y'', \ldots]$. I is a *differential ideal* iff derivation does not lead out of I; i.e., if $F \in I$ then also $F' \in I$.

In this chapter we deal with the problem of deciding the existence of rational solutions of AODEs of order 1. If the AODE has rational solutions, we want to find a rational general solution; i.e., a rational solution containing a transcendental constant c.

The radical differential ideal $\{F\}$ can be decomposed

$$\{F\} = \underbrace{(\{F\} : S)}_{general\ component} \cap \underbrace{\{F, S\}}_{singular\ component} ,$$

where S is the separant of F (derivative of F w.r.t. $y^{(n)}$).
If F is irreducible, the general component $\{F\} : S$ is a prime differential ideal; its generic zero is called a *general solution* of the AODE $F(x, y, y', \ldots, y^{(n)}) = 0$. (compare [Ritt50])

Problem: Rational general solution of AODE of order 1

given: an AODE $F(x, y, y') = 0$, F irreducible in $\overline{\mathbb{Q}}[x, y, y']$

decide: does this AODE have a rational general solution

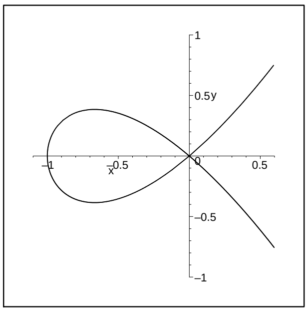
find: if so, find it

Example: $F \equiv y'^2 + 3y' - 2y - 3x = 0$.
general solution: $y = \frac{1}{2}((x + c)^2 + 3c)$, where c is an arbitrary constant.
The separant of F is $S = 2y' + 3$. So the singular solution of F is $y = -\frac{3}{2}x - \frac{9}{8}$.

We will solve the problem by rationally parametrizing the plane algebraic curve defined by $F(u, v) = 0$. So let us briefly recall some facts about such parametrizations.

An *algebraic variety* $\mathcal{V}$ is the zero locus of a (finite) set of polynomials F, or of the ideal $I = \langle F \rangle$.
A *rational parametrization* of $\mathcal{V}$ is a rational map $\mathcal{P}$ from a full (affine, projective) space covering $\mathcal{V}$; i.e. $\mathcal{V} = \overline{\mathrm{im}(\mathcal{P})}$ (Zariski closure).
A variety having a rational parametrization is called *unirational*; and *rational* if $\mathcal{P}$ has a rational inverse.

The singular cubic $y^2 - x^3 - x^2 = 0$



has the rational, in fact polynomial, parametrization $x(t) = t^2 - 1, \quad y(t) = t^3 - t$. So this is a unirational curve.

Some facts:

- a parametrization of a variety is a *generic point* or *generic zero* of the variety; i.e. a polynomial vanishes on the variety if and only if it vanishes on this generic point
- so only irreducible varieties can be rational
- a rationally invertible parametrization $\mathcal{P}$ is called a *proper* parametrization; every rational curve or surface has a proper parametrization (Lüroth, Castelnuovo); but not so in higher dimensions

For details on parametrizations of algebraic curves we refer to

- J.R. Sendra, F. Winkler, S. Pérez-Díaz,
  Rational Algebraic Curves – A Computer Algebra Approach,
  Springer-Verlag Heidelberg (2008)

The algebro-geometric method:

- A rational solution of $F(y, y') = 0$ corresponds to a proper (because of the degree bounds) rational parametrization of the algebraic curve $F(y, z) = 0$.
- Conversely, from a proper rational parametrization $(f(x), g(x))$ of the curve $F(y, z) = 0$ we get a rational solution of $F(y, y') = 0$ if and only if there is a linear rational function $T(x)$ such that $f(T(x))' = g(T(x))$.
  If $T(x)$ exists, then a rational solution of $F(y, y') = 0$ is: $y = f(T(x))$.
  The rational general solution of $F(y, y') = 0$ is (for an arbitrary constant c): $y = f(T(x + c))$

Feng and Gao (in [FeG06] and also in their paper in ISSAC'2004) described a complete algorithm along these lines

- R. Feng, X-S. Gao,
  "Rational general solutions of algebraic ordinary differential equations",
  Proc. ISSAC2004. ACM Press, New York, 155-162, 2004.
- R. Feng, X-S. Gao,
  "A polynomial time algorithm for finding rational general solutions of first order autonomous ODEs",
  J. Symb. Comp., 41, 739-762, 2006.

This is based on degree bounds derived in

- J.R. Sendra, F. Winkler,
  "Tracing index of rational curve parametrizations",
  Computer Aided Geometric Design, 18:771–795, 2001.