Unification by Narrowing

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May 20, 2014
What is narrowing?

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  - sound and complete method for solving $E$-unification problems in theories presented by complete term rewriting systems.
  - computational model for functional logic programming (FLP)
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- **Functional logic programming** = programming style resulted by the integration of two declarative programming styles: Functional Programming and Logic Programming
  - \(FLP = FP + LP\)

- **Functional Programming**
  - Program = term rewriting system (usually terminating and confluent)
  - Computation = reduction to normal form
  - \(\Rightarrow \text{value}\)
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  \[
  \text{FLP} = \text{FP} + \text{LP}
  \]

- **Functional Programming**
  - Program = term rewriting system
    (usually terminating and confluent)
  - Computation = reduction to normal form
    \(\Rightarrow\) value

- **Logic Programming**
  - Program = set of Horn clauses (rules and facts)
  - Computation: SLD resolution of goals
    \(\Rightarrow\) computed answers
**DESIRE**: inherit the best features from both logic programming and functional programming

- **Advantages of logic programming:**
  - Logical variables; sound and complete search strategy for answers to queries

- **Advantages of functional programming:**
  - More efficient operational behaviour: evaluation of function calls is more deterministic than computing answers to queries.
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**Approaches to integrate FP with LP and define FLP=FP+LP**

1. Integrate functions into LP.
2. Extend FP with equational queries involving function calls and logical variables.
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Approaches to integrate FP with LP and define FLP=FP+LP

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Historically, both approaches resulted in languages with similar computational models.
Basic notions
Rewrite rules as directed equations

Starting from

\[ f, g, h, \ldots \in \mathcal{F} : \text{ranked signature of function symbols;} \]
\[ ar(f) \in \mathbb{N} \text{ for all } f \in \mathcal{F} \]
\[ x, y, z, \ldots \in \mathcal{V} : \text{countable set of variables} \]

we build

- **Terms:** \( t \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \): \( t ::= x \mid f(t_1, \ldots, t_n) \) where \( ar(f) = n \)

Convention: abbreviate \( f() \) by \( f \)

- **Equations:** \( e ::= s = t \) where \( s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \)

- **Rewrite rules:** \( l \rightarrow r \) where \( l, r \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \), \( l \notin \mathcal{V} \), \( \text{vars}(r) \subseteq \text{vars}(l) \). A TRS is a set of rewrite rules.

- **Rewriting** with a TRS \( \mathcal{R} = \) replacing “equals by equals” in a directed manner: \( s \rightarrow_{\mathcal{R}} t \) if there exist \( p \in \text{Pos}(s) \), \( (l \rightarrow r) \in \mathcal{R} \), and substitution \( \sigma : \mathcal{V} \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{V}) \) such that \( s|_p = l\sigma \) and \( t = s[r\sigma]|_p \).
Equational reasoning = reasoning with equations in the quotient algebra $\mathcal{T}(\mathcal{F}, \mathcal{V})/\equiv_E$ where $\equiv_E$ is the congruence relation induced on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ by a set of equations $E$ (the equational axioms);

$\equiv_E$ is the least equivalence relation on $\mathcal{T}(\mathcal{F}, \mathcal{V})$, which satisfies the following two additional conditions:

**Substitution:** if $s \equiv_E t$ then $s\sigma \equiv_E t\sigma$ for all substitutions $\sigma$.

**Replacement:** if $l \equiv_E r$ and $s|_{\rho} = l$ then $s \equiv_E s[r]_{\rho}$.
Equational reasoning

Equational reasoning = reasoning with equations in the quotient algebra $\mathcal{T}(\mathcal{F}, \mathcal{V})/\equiv_E$ where $\equiv_E$ is the congruence relation induced on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ by a set of equations $E$ (the equational axioms);

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Replacement: if $l \equiv_E r$ and $s|_p = l$ then $s \equiv_E s[r]_p$.

$E$-unification problem

Given a set of equations $E$ and a system of equations

$$\Gamma : s_1 = t_1, \ldots, s_n = t_n$$

Find a representation of the set of substitutions $\sigma$ such that $s_i\sigma \equiv_E t_i\sigma$ for all $i = 1..n$.

$\Gamma$ is an $E$-unification problem, and a $\sigma$ is a unifier of $\Gamma$. 

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Unification by Narrowing
**The unification hierarchy (1)**

**Assumptions:**

- $E$: set of equations
- $\Gamma$: $E$-unification problem
- $\text{Sol}(\Gamma)$: the set of all unifiers of $\Gamma$

- $E$ induces an order on terms: $s \leq_E t$ if $s\sigma =_E t$ for some $\sigma$.
- A set $S$ of substitutions is a complete set of unifiers (csu) of $\Gamma$ if
  1. $S \subseteq \text{Sol}(\Gamma)$
  2. For any $\theta \in \text{Sol}(\Gamma)$ there is a $\sigma \in S$ such that $\sigma(x) \leq_E \theta(x)$ for all $x \in \text{vars}(\Gamma)$

$S$ is a minimal csu (mcsu) of $\Gamma$ if it also satisfies the following condition:

- If $\sigma_1, \sigma_2 \in S$ and $\sigma_1(x) \leq_E \sigma_2(x)$ for all $x \in \text{vars}(\Gamma)$, then $\sigma_1 = \sigma_2$. 
mcsu of $\Gamma$ may not exist!

**Unification problem without mcsu [Schmidt-Schauss, 1986]**

$$E = \{ f(f(x, y), z) = f(x, f(y, z)), f(x, x) = x \}$$

$$\Gamma : f(z, f(a, f(x, f(a, z)))) = f(z, f(a, z))$$

[Siekmann, 1978] introduced the following hierarchy of unification problems:

- **unitary**: they have a mcsu with 0 or 1 elements.
- **finitary**: they have a mcsu with finite number of elements.
- **infinitary**: they have a mcsu with infinite number of elements.
- **nullary**: they do not have mcsu.
mcsu of $\Gamma$ may not exist!

Unification problem without mcsu [Schmidt-Schauss, 1986]

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[Siekmann, 1978] introduced the following hierarchy of unification problems:

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- **nullary**: they do not have mcsu.

[Nutt, 1991] proved that the unification hierarchy is undecidable.
**Implicit Assumptions:** $\mathcal{R}$ is a TRS, and

- $=_{\mathcal{R}}$ is the congruence relation induced by $\mathcal{R}$, viewed as system of equations.
- $s \downarrow_{\mathcal{R}} t : \overset{\text{def}}{\iff} \text{there exists } u \text{ s.t. } s \rightarrow^*_{\mathcal{R}} u \text{ and } t \rightarrow^*_{\mathcal{R}} u$.

From now on we will consider systems of equations (also known as goals)

$$\Gamma : s_1 = t_1, \ldots, s_n = t_n$$

interpreted in equational theories presented by term rewriting systems. This means that:

- We interpret the equality $=$ as $=_{\mathcal{R}}$. If $\mathcal{R}$ is confluent then $=_{\mathcal{R}}$ coincides with $\downarrow_{\mathcal{R}}$.
- We wish to compute a compute set of $\mathcal{R}$-unifiers of $\Gamma$. These $\mathcal{R}$-unifiers are also known as solutions of $\Gamma$. 

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**Unification by Narrowing**
A TRS $\mathcal{R}$ is

- **terminating** (or **normalizing**) if very sequence of rewrite steps will eventually terminate: $t \rightarrow^*_{\mathcal{R}} t_1 \rightarrow^*_{\mathcal{R}} \ldots \rightarrow^*_{\mathcal{R}} t_n \not\rightarrow^*_{\mathcal{R}} t_n$ is called a **normal form** of $t$.

- **weakly-normalizing** if for any term $t$ there exists a rewrite termination that ends with a normal form:
  
  $t = t_0 \rightarrow^*_{\mathcal{R}} t_1 \rightarrow^*_{\mathcal{R}} \ldots \rightarrow^*_{\mathcal{R}} t_n \not\rightarrow^*_{\mathcal{R}}$

- **confluent** if $t_1 \downarrow^*_{\mathcal{R}} t_2$ whenever $t \rightarrow^*_{\mathcal{R}} t_1$ and $t \rightarrow^*_{\mathcal{R}} t_2$.

- **semi-complete** if it is weakly-normalizing and confluent.

- **complete** if it is terminating and confluent.
A TRS $\mathcal{R}$ is

- **terminating** (or **normalizing**) if very sequence of rewrite steps will eventually terminate: $t \rightarrow_{\mathcal{R}} t_1 \rightarrow_{\mathcal{R}} \ldots \rightarrow_{\mathcal{R}} t_n \not\rightarrow_{\mathcal{R}}$
  $t_n$ is called a normal form of $t$.

- **weakly-normalizing** if for any term $t$ there exists a rewrite termination that ends with a normal form:
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- **confluent** if $t_1 \downarrow_{\mathcal{R}} t_2$ whenever $t \rightarrow^{*}_{\mathcal{R}} t_1$ and $t \rightarrow^{*}_{\mathcal{R}} t_2$.

- **semi-complete** if it is weakly-normalizing and confluent.

- **complete** if it is terminating and confluent.

**Remarks**

- If $\mathcal{R}$ is confluent then $s =_{\mathcal{R}} t$ iff $s \downarrow_{\mathcal{R}} t$.

- If $\mathcal{R}$ is complete then $=_{\mathcal{R}}$ is decidable.
Program = complete TRS defined over a signature
\( \mathcal{F} = \mathcal{F}_d \uplus \mathcal{F}_c \) where
- \( \mathcal{F}_d \): set of defined function symbols
- \( \mathcal{F}_c \): set of constructors

Rewrite rules are of the form \( f(s_1, \ldots, s_n) \rightarrow t \) where
\( f \in \mathcal{F}_d \) and \( s_1, \ldots, s_n \in \mathcal{T}(\mathcal{F}_c, \mathcal{V}) \).

Computation in FP: computes the (unique) normal form of a term \( t \)
- Strict languages: terms are reduced by leftmost innermost rewriting.
- Lazy languages: terms are reduced by leftmost outermost rewriting.

Computation in FLP: find a csu (preferably mcsu) of
\[
\Gamma : s_1 = t_1, \ldots, s_n = t_n
\]
Theoretical results

Narrowing

**Assumption:** \( \mathcal{R} \) is a TRS.

**Definition (Fresh variant)**

A fresh variant of a rewrite rule \( l \rightarrow r \) is a bijective substitution \( \sigma \) with \( \text{dom}(\sigma) = \text{vars}(l) \) and \( \sigma(x) \) is a fresh new variable for each \( x \in \text{dom}(\sigma) \).

**Definition (Narrowing [Slagle, 1974])**

\( s \) is narrowable into \( t \), notation \( s \rightleftharpoons_{\sigma, \mathcal{R}} t \), if there exist

- a narrowing position \( p \in \text{Pos}(s) \) such that \( s|_{p} \notin \mathcal{V} \)
- a fresh variant \( l \rightarrow r \) of a rewrite rule of \( \mathcal{R} \)

such that \( \sigma = \text{mgu}(s|_{p}, l) \) and \( t = s[r]_{p}\sigma \).

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Unification by Narrowing
**ASSUMPTION:** $\mathcal{R}$ is a TRS.

**Definition (Fresh variant)**

A fresh variant of a rewrite rule $l \rightarrow r$ is a bijective substitution $\sigma$ with $\text{dom}(\sigma) = \text{vars}(l)$ and $\sigma(x)$ is a fresh new variable for each $x \in \text{dom}(\sigma)$.

**Definition (Narrowing [Slagle, 1974])**

$s$ is narrowable into $t$, notation $s \rightsquigarrow_{\sigma, \mathcal{R}} t$, if there exist
- a narrowing position $p \in \text{Pos}(s)$ such that $s|_p \notin \mathcal{V}$
- a fresh variant $l \rightarrow r$ of a rewrite rule of $\mathcal{R}$ such that $\sigma = \text{mgu}(s|_p, l)$ and $t = s[r]_p\sigma$.

**NOTATION:** A derivation $t_0 \rightsquigarrow_{\sigma_1, \mathcal{R}} t_1 \rightsquigarrow_{\sigma_2, \mathcal{R}} \cdots \rightsquigarrow_{\sigma_n, \mathcal{R}} t_n$ is abbreviated $t_0 \rightsquigarrow^*_{\mathcal{R}} t_n$, or simply $t_0 \rightsquigarrow^*_{\sigma} t_n$, where $\sigma = \sigma_1 \ldots \sigma_n$. 
Theorem ([Hullot, 1985])

If \( R \) is complete then

**Soundness:** If \( s = t \leadsto_\sigma s' = t' \) and \( \theta = \text{mgu}(s', t') \) then
\[
(s\sigma\theta) =_R (t\sigma\theta)
\]

**Completeness:** If \( s\theta =_R t\theta \) then there exist

- \( s = t \leadsto_\sigma s' = t' \) and
- \( \sigma' \in \text{mgu}(s', t') \)

such that \( \sigma\sigma' \leq_R \theta \ [\text{vars}(s, t)] \).
Narrowing
Main properties

Theorem ([Hullot, 1985])

If $\mathcal{R}$ is complete then

**Soundness:** If $s \vdash_{\sigma} s' = t'$ and $\theta = \text{mgu}(s', t')$ then

$$ (s\sigma\theta) =_{\mathcal{R}} (t\sigma\theta) $$

**Completeness:** If $s\theta =_{\mathcal{R}} t\theta$ then there exist

- $s = t \vdash_{\sigma}^* s' = t'$ and
- $\sigma' \in \text{mgu}(s', t')$

such that $\sigma\sigma' \leq_{\mathcal{R}} \theta [\text{vars}(s, t)]$.

**Question:** Can we drop the condition of termination of $\mathcal{R}$, and still have soundness and completeness?

**Answer:** Yes, if we restrict ourselves to normalized unifiers: $NSol(\Gamma) = \{ \theta \in Sol(\Gamma) \mid x\theta \text{ is normal form, for all } x \in \text{dom}(\theta) \}$. 
Narrowing computations

Example

\[ \mathcal{R} = \{0 + x \rightarrow x, s(x) + y \rightarrow s(x + y), x = x \rightarrow \text{true}\}. \]

Let's solve \( z + z = s(s(0)) \):

\[
\begin{align*}
    z + z & = s(s(0)) \quad \leadsto \quad \{y_1 \mapsto s(x_1), z \mapsto s(x_1)\}, s(x_1) + y_1 \rightarrow s(x_1 + y_1) \\
    s(x_1 + s(x_1)) & = (s(0)) \quad \leadsto \quad \{x_1 \mapsto 0, x_2 \mapsto s(0)\}, 0 + x_2 \mapsto x_2 \\
    s(s(0)) & = s(s(0)) \quad \leadsto \quad \{x_3 \mapsto s(s(0))\}, x_3 + x_3 \mapsto \text{true} \text{ true}
\end{align*}
\]

Solution: \( \{y_1 \mapsto s(x_1), z \mapsto s(x_1)\}\{x_1 \mapsto 0, x_2 \mapsto s(0)\}\{x_3 \mapsto s(s(0))\} \)

\(= \{y_1 \mapsto s(0), z \mapsto s(0), x_1 \mapsto 0, x_2 \mapsto s(0), x_3 \mapsto s(s(0))\} \)

restricted to \( \text{vars}(z + z = 0) = \{z\} \), is \( \theta = \{z \mapsto s(0)\} \)

There are also several failed attempts to compute \( \mathcal{R} \)-unifiers:

\[
\begin{align*}
    z + z & = s(s(0)) \quad \leadsto \quad \{z \mapsto 0, x_1 \mapsto 0\}, 0 + x_1 \mapsto x_1 \quad 0 = s(s(0)) \not\rightarrow
\end{align*}
\]
Let $\mathcal{R}$ be a confluent TRS, $\Gamma : s_1 = t_1, \ldots, s_n = t_n$, and

- $\mathcal{R}_+ := \mathcal{R} \cup \{(x = x) \rightarrow \text{true}\}$
- $\top := \text{generic notation for system containing only true-s}$

**Definition**

$\leadsto_\mathcal{R}$ is extended to act on systems of equations as follows:

$\Gamma_1, e, \Gamma_2 \leadsto_{\sigma, \mathcal{R}} (\Gamma_1, e', \Gamma_2)_{\sigma}$

if $e \leadsto_{\sigma, \mathcal{R}} e'$ where $e$ is a non-true equation.

**NOTATION:** Like before, we abbreviate $\Gamma_0 \leadsto_{\sigma_1} \ldots \leadsto_{\sigma_n} \Gamma_n$ with $\Gamma_0 \leadsto^*_{\sigma} \Gamma_n$, where $\sigma = \sigma_1 \ldots \sigma_n$. Also, we define the set of answers computed by narrowing: $\text{Ans}(\Gamma) = \{ \sigma \mid \Gamma \leadsto^*_{\sigma} \top \}$

**Corollary**

$\text{Ans}(\Gamma)$ is a csu of $\Gamma$. 
The computation of $\text{Ans}(\Gamma)$ is highly nondeterministic, due to the selection of

1. the narrowing position
2. the rewrite rule to be applied at the narrowing position

A more deterministic version of narrowing, still sound and complete w.r.t. normalized unifiers, is basic narrowing ([Hullot, 1987], [Middeldorp et al, 1996])

**Definition (Position constraint)**

A position constraint for $\Gamma$ is a mapping that assigns to every equation $e \in \Gamma$ a subset of $\text{Pos}_F(e) = \{p \in \text{Pos}(e) \mid e|_p \notin \mathcal{V}\}$. The position constraint that assigns to every $e \in \Gamma$ the set $\text{Pos}_F(e)$ is denoted by $\Gamma$. 

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Unification by Narrowing
Definition (Basic derivation)

\( \Gamma_1 \rightsquigarrow_{\sigma_1, e_1, p_1, l_1 \rightarrow r_1 \ldots} \rightsquigarrow_{\sigma_{n-1}, e_{n-1}, p_{n-1}, l_{n-1} \rightarrow r_{n-1}} \Gamma_n \) is based on a position constraint \( B_1 \) for \( \Gamma_1 \) if \( p_i \in B_i(e_i) \) for \( 1 \leq i \leq n - 1 \), where

\[
B_{i+1}(e) := \begin{cases} 
B_i(e') & \text{if } e' \in \Gamma_i \setminus \{e_i\} \\
B(B_i(e_i), p_i, r_i) & \text{if } e' = e_i[r_i|p_i]
\end{cases}
\]

for all \( 1 \leq i < n - 1 \) and \( e = e'\sigma_i \in \Gamma_{i+1} \), with \( B(B_i(e_i), p_i, r_i) \) abbreviating the set of positions

\[
B_i(e_i) \setminus \{q \in B_i(e_i) \mid q \geq p_i\} \cup \{p_i \cdot q \in Pos_F(e) \mid q \in Pos_F(r_i)\}.
\]

Such a narrowing derivation of \( \Gamma_1 \) is basic if \( B_1 = \Gamma_1 \).
Remark: In a basic narrowing derivation, narrowing is never applied to a subterm introduced by a previous narrowing substitution.

Theorem ([Hullot, 1987], [Middeldorp and Hamoen, 1994])

Let $\mathcal{R}$ be a confluent TRS and $\Gamma$ a system of equations. For every normalized unifier $\theta$ of $G$ there exists a basic narrowing refutation $\Gamma \leadsto^* \sigma$ such that $\sigma \leq_{\mathcal{R}} \theta [\text{vars}(\Gamma)]$ provided one of the following conditions is satisfied:

1. $\mathcal{R}$ is terminating
2. $\mathcal{R}$ is orthogonal and $\Gamma \theta$ has an $\mathcal{R}$-normal form
3. $\mathcal{R}$ is right-linear
\[ \mathcal{R} = \{rev(rev(x)) \rightarrow x\} \] specifies a property of the reverse operation on lists.

- An infinite non-basic narrowing derivation

\[ \Gamma : rev(x) = x \quad \leadsto \{x \mapsto rev(x_1)\}, 1, rev(rev(x_1)) \rightarrow x_1 \quad x_1 = rev(x_1) \]
\[ \leadsto \{x_1 \mapsto rev(x_2), 2, rev(rev(x_2)) \rightarrow x_2\} \quad rev(x_2) = x_2 \]
\[ \leadsto \{x_2 \mapsto rev(x_3), 1, rev(rev(x_3)) \rightarrow x_3\} \quad \cdots \]

- The only basic narrowing derivation of the same \( \Gamma \) is

\[ \Gamma : rev(x) = x \quad \leadsto \{x \mapsto rev(x_1)\}, 1, rev(rev(x_1)) \rightarrow x_1 \quad x_1 = rev(x_1) \]

Basic narrowing prohibits any further narrowing steps \( \Rightarrow \) \( \Gamma \) has no unifiers.
Basic narrowing
Other useful properties

Theorem ([Hullot, 1980])

If $\mathcal{R} = \{ l_i \rightarrow r_i \mid 1 \leq i \leq n \}$ is a complete TRS, and any basic narrowing derivation starting from $r_i$ terminates, then all basic narrowing derivations starting from any term terminate.
Basic narrowing
Other useful properties

Theorem ([Hullot, 1980])

If $\mathcal{R} = \{ l_i \rightarrow r_i \mid 1 \leq i \leq n \}$ is a complete TRS, and any basic narrowing derivation starting from $r_i$ terminates, then all basic narrowing derivations starting from any term terminate.

Corollary

Basic narrowing becomes a decision procedure for E-unification if the conditions of the previous theorem hold.
Narrowing calculi

- Computational model of several functional logic programming languages.
- Narrowing is a complicated operation \(\Rightarrow\) various narrowing calculi consisting of more elementary inference rules that simulate narrowing have been proposed.

Properties of narrowing calculi

- Easier to analyse than the narrowing operation.
- Three sources of nondeterminism, due to the choice of:
  1. the equation of the system
  2. the inference rule to be applied
  3. the rewrite rule of the TRS (for certain inference rules)

- Several criteria have been proposed to reduce these sources of nondeterminism under reasonable assumptions.
Lazy narrowing calculi
LNC [Middeldorp and Okui, 1999]

[o] outermost narrowing: \( \frac{\Gamma_1, f(s_1, \ldots, s_n) \simeq t, \Gamma_2}{\Gamma_1, s_1 = l_1, \ldots, s_n = l_n, r = t, \Gamma_2} \)
if \( f(l_1, \ldots, l_n) \rightarrow r \) is a fresh variant of a rule from \( \mathcal{R} \)

[i] imitation: \( \frac{\Gamma_1, f(s_1, \ldots, s_n) \simeq x, \Gamma_2}{(\Gamma_1, s_1 = x_1, \ldots, s_n = x_n, \Gamma_2)\theta} \)
if \( \theta = \{ x \mapsto f(x_1, \ldots, x_n) \} \) with \( x_1, \ldots, x_n \) fresh variables.

[d] decomposition: \( \frac{\Gamma_1, f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n), \Gamma_2}{\Gamma_1, s_1 = t_1, \ldots, s_n = t_n, \Gamma_2} \)

[v] variable elimination: \( \frac{\Gamma_1, x \simeq t, \Gamma_2}{(\Gamma_1, \Gamma_2)\sigma} \)
if \( x \not\in \text{vars}(t) \) and \( \sigma = \{ x \mapsto t \} \)

[t] removal of trivial equations: \( \frac{\Gamma_1, x = x, \Gamma_2}{\Gamma_1, \Gamma_2} \)

The red equations produced by [o] are called parameter-passing equations.
NOTATION:

- \( \Gamma \Rightarrow [\alpha], \sigma \Gamma' \) if \( \Gamma \) and \( \Gamma' \) are the upper and lower parts of an inference rule \( [\alpha] \) (\( \alpha \in \{o, i, d, v, t\} \)) and \( \sigma \) is the substitution computed by that inference rule.

- \( \square \) denotes the system with no equations.

- An LNC-derivation \( \Gamma_0 \Rightarrow [\alpha_1], \sigma_1 \ldots \Rightarrow [\alpha_n], \sigma_n \Gamma_n \) is abbreviated \( \Gamma_0 \Rightarrow^* \square \) where \( \sigma = \sigma_1 \ldots \sigma_n \).
Theorem

If $\mathcal{R}$ is confluent and $\theta$ is a normalized $\mathcal{R}$-unifier of $\Gamma$ then there exists $\Gamma \Rightarrow^*_\sigma \Box$ respecting leftmost equation selection strategy such that $\sigma \leq \theta [\text{vars}(\Gamma)]$.
Theorem

If $\mathcal{R}$ is confluent and $\theta$ is a normalized $\mathcal{R}$-unifier of $\Gamma$ then there exists $\Gamma \Rightarrow^*_\sigma \Box$ respecting leftmost equation selection strategy such that $\sigma \leq \theta [\text{vars}(\Gamma)]$.

Theorem

Let $\mathcal{R}$ be a confluent TRS, $\Gamma$ a system of equations, and $S$ any selection function for equations from a system. For every normalized solution $\theta$ of $\Gamma$ there exists an LNC-refutation $\Gamma \Rightarrow^*_\sigma \Box$ respecting $S$ such that $\sigma \leq \theta [\text{vars}(\Gamma)]$ provided one of the following conditions holds:

1. $\mathcal{R}$ is terminating
2. $\mathcal{R}$ is orthogonal and $\Gamma \theta$ has an $\mathcal{R}$-normal form
3. $\mathcal{R}$ is right-linear
Refinement of LNC which performs **eager** variable elimination for descendants of parameter-passing equations:

- Whenever we select an equation $x \simeq t$ with $x \notin \text{vars}(t)$, which is descendant of a parameter-passing equation, we apply inference rule $[\nu]$.

**Theorem**

Let $\mathcal{R}$ be an orthogonal TRS and $\Gamma$ a system of equations. For every $\mathcal{R}$-normalized unifier $\theta$ of $\Gamma$ there exists an eager LNC-refutation $G \Rightarrow^* \square$ respecting leftmost equation selection strategy, such that $\sigma \leq \theta [\text{vars}(\Gamma)]$.

Note that a TRS $\mathcal{R}$ is **orthogonal** if

1. It is left-linear, i.e., no equation appears twice in any lhs of some rewrite rule
2. It’s rewrite rules are non-overlapping
Refinements of LNC
LNC\(_d\) [Middeldorp and Okui, 1999]

Designed for strict solving of systems of equations

**Definition**

Let \( \mathcal{R} \) be a TRS. A substitution \( \sigma \) is a strict solution of a system \( \Gamma \) if for every equation \( s = t \) in \( \Gamma \) there exists a constructor term \( u \) such that \( s\sigma \rightarrow^* R u \) and \( t\sigma \rightarrow^* R u \).

LNC\(_d\) is a refinement of calculus LNC which distinguishes:

- \( \mathcal{F} = \mathcal{F}_c \uplus \mathcal{F}_d \), where
  \[ \mathcal{F}_d := \{ f \in \mathcal{F} \mid \exists (f(s_1, \ldots, s_n) \rightarrow r) \in \mathcal{R} \} \text{ and } \mathcal{F}_c = \mathcal{F} \setminus \mathcal{F}_d \]
- Descendants of initial equations (written as \( s \equiv t \)) from descendants of parameter-passing equations (written as \( s \triangleright t \)). We write \( s \cong t \) if \( s \equiv t \) or \( t \equiv s \)
Inference rules for initial equations

\[ o \] ≡ outermost narrowing: \[
\begin{align*}
    f(s_1, \ldots, s_n) & \equiv t, \Gamma \\
    s_1 & \triangleright l_1, \ldots, s_n & \triangleright l_n, r & \equiv t, \Gamma
\end{align*}
\]
if root(t) \( \notin \mathcal{F}_d \) and \( f(s_1, \ldots, s_n) \to r \) is a fresh variant of a rewrite rule from \( \mathcal{R} \).

\[ i \] ≡ imitation: \[
\begin{align*}
    f(s_1, \ldots, s_n) & \equiv x, \Gamma \\
    (s_1 & \equiv x_1, \ldots, s_n & \equiv x_n, \Gamma)\sigma
\end{align*}
\]
if \( f \in \mathcal{F}_c, c \notin \text{vars}_c(f(s_1, \ldots, s_n)), f(s_1, \ldots, s_n) & \notin \mathcal{T}(\mathcal{F}_c, \mathcal{V}) \), and \( \sigma = \{ x \mapsto f(x_1, \ldots, x_n) \} \) with \( x_1, \ldots, x_n \) fresh variables.

\[ d \] ≡ decomposition: \[
\begin{align*}
    f(s_1, \ldots, s_n) & \equiv f(t_1, \ldots, t_n), \Gamma \\
    s_1 & \equiv t_1, \ldots, s_n & \equiv t_n, \Gamma
\end{align*}
\]

\[ v \] ≡ variable elimination: \[
\begin{align*}
    s & \equiv x, \Gamma \\
    \Gamma\sigma
\end{align*}
\]
if \( x \notin \text{vars}(s) \) and \( \sigma = \{ x \mapsto s \} \).

\[ t \] ≡ removal of trivial equations: \[
\begin{align*}
    x & \equiv x, \Gamma \\
    \Gamma
\end{align*}
\]
Inference rules for descendants of parameter-passing equations

\[ [\sigma] \triangleright \text{outermost narrowing:} \]
\[
\frac{f(s_1, \ldots, s_n) \triangleright \Gamma}{s_1 \triangleright l_1, \ldots, s_n \triangleright l_n, r \triangleright t, \Gamma}
\]
if \( \text{root}(t) \not\in \mathcal{F}_d \) and \( f(s_1, \ldots, s_n) \rightarrow r \) is a fresh variant of a rewrite rule from \( \mathcal{R} \)

\[ [d] \triangleright \text{decomposition:} \]
\[
\frac{f(s_1, \ldots, s_n) \triangleright f(t_1, \ldots, t_n), \Gamma}{s_1 \triangleright t_1, \ldots, s_n \triangleright t_n, \Gamma}
\]

\[ [v] \triangleright \text{variable elimination:} \]
\[
\frac{s \triangleright x, \Gamma}{\Gamma \sigma} \quad \frac{x \triangleright s, \Gamma}{\Gamma \sigma}
\]
if \( x \not\in \text{vars}(s) \) and \( \sigma = \{ x \mapsto s \} \)
Theorem

Let $\mathcal{R}$ be a left-linear confluent constructor system and $\Gamma$ a system of equations. For every strict and normalized solution $\theta$ of $\Gamma$ there exists an $\text{LNC}_d$-refutation $G \Rightarrow^* \Box$ such that $\sigma \leq_{\mathcal{R}} \theta \ [\text{vars}(\Gamma)]$
Theorem

Let $\mathcal{R}$ be a left-linear confluent constructor system and $\Gamma$ a system of equations. For every strict and normalized solution $\theta$ of $\Gamma$ there exists an LNC$_d$-refutation $G \Rightarrow^* \Box$ such that $\sigma \leq_{\mathcal{R}} \theta \ [\text{vars}(\Gamma)]$

Remark

The only source of nondeterminism of the lazy narrowing calculus LNC$_d$ is the choice of the rewrite rule when applying inference rules $[o]_\equiv$ and $[o]_>$. The other sources of nondeterminism disappeared:

1. The selected equation is always the leftmost
2. There is only at most one applicable inference rule.
A conditional TRS (CTRS) consists of conditional rewrite rules 
\( l \rightarrow r \leftarrow c \) where the conditional part is a (possibly empty) sequence 
\( s_1 = t_1, \ldots, s_n = t_n \) of equations. We require \( l \not\in \mathcal{V} \).

\[
e\text{vars}(l \rightarrow r \leftarrow c) := \text{vars}(r, c) \setminus \text{vars}(l)
\]
A conditional TRS (CTRS) consists of conditional rewrite rules \( l \rightarrow r \leftarrow c \) where the conditional part is a (possibly empty) sequence \( s_1 = t_1, \ldots, s_n = t_n \) of equations. We require \( l \not\in \mathcal{V} \).

\[
evars(l \rightarrow r \leftarrow c) := \text{vars}(r, c) \setminus \text{vars}(l)
\]

CTRSs are classified according to the distribution of variables in rewrite rules, into:

1-CTRS \( \text{vars}(r, c) \subseteq \text{vars}(l) \) for all rules \( l \rightarrow r \leftarrow c \)

2-CTRS \( \text{vars}(r) \subseteq \text{vars}(l) \) for all rules \( l \rightarrow r \leftarrow c \)

3-CTRS \( \text{vars}(r) \subseteq \text{vars}(l, c) \) for all rules \( l \rightarrow r \leftarrow c \)

**Remark:** Extra variables enable a note natural style of writing program specifications

**Example (Fibonacci numbers)**

\[
0 + y \rightarrow y, \ s(x) + y \rightarrow s(x + y), \ 
\]

\[
\text{fib}(0) \rightarrow \langle 0, s(0) \rangle, \ 
\]

\[
\text{fib}(s(x)) \rightarrow \langle z, y + z \rangle \leftarrow \text{fib}(x) = \langle y, z \rangle
\]
ASSUMPTIONS:

- every CTRS $\mathcal{R}$ contains the rewrite rule $x = x \rightarrow \text{true}
- \text{true and } = \text{ do not occur in other rewrite rules of } \mathcal{R}
- \top \text{ denotes any sequence of } \text{trues}

We define inductively the unconditional TRSs $\mathcal{R}_n$ for $n \geq 0$:

$$\mathcal{R}_0 := \{x = x \rightarrow \text{true}\}$$

$$\mathcal{R}_{n+1} := \{l\sigma \rightarrow r\sigma \mid l \rightarrow r \leftarrow c \in \mathcal{R} \text{ and } c\sigma \rightarrow^*_n \top\}$$

and abbreviate $\rightarrow \mathcal{R}_n$ by $\rightarrow_n$
ASSUMPTIONS:
- every CTRS $R$ contains the rewrite rule $x = x \rightarrow true$
- $true$ and $=$ do not occur in other rewrite rules of $R$
- $\top$ denotes any sequence of $true$ s

We define inductively the unconditional TRSs $R_n$ for $n \geq 0$:

$$R_0 := \{ x = x \rightarrow true \}$$

$$R_{n+1} := \{ l\sigma \rightarrow r\sigma \mid l \rightarrow r \leftarrow c \in R \text{ and } c\sigma \rightarrow^{*}_{R_n} \top \}$$

and abbreviate $\rightarrow_{R_n}$ by $\rightarrow_n$

Remarks

We interpret equality as joinability; such kind of CTRSs are known as join CTRSs in the literature.
Level confluence: \( \mathcal{R} \) is level-confluent if every \( \mathcal{R}_n \) is confluent.

Shallow-confluence: \( \mathcal{R} \) is shallow-confluent if
\[
^*_m \leftarrow \circ \rightarrow^*_n \subseteq^* n \leftarrow \circ \rightarrow^*_m \text{ for all } m, n \geq 0.
\]

Decreasingness: \( \mathcal{R} \) is decreasing if there exists a well-founded \( \succ \) order on \( \mathcal{T}(\mathcal{F}, \mathcal{V}) \) with the following properties:
- \( \succ \) contains \( \rightarrow_{\mathcal{R}} \)
- \( \succ \) has the subterm property (i.e., \( \prec \subseteq \succ \) where \( s \succ t \) iff \( t \) is a proper subterm of \( s \))
- \( t\sigma \succ s\sigma \) and \( s\sigma \succ t\sigma \) for every \( l \rightarrow r \leftarrow c \in \mathcal{R} \), every \( s = t \) from \( c \), and every substitution \( \sigma \).

Remark
Shallow-confluent CTRSs are level-confluent, but the reverse is not true.
ASSUMPTION: $R$ is a CTRS

$$\frac{(\Gamma', e[r]_p, c, \Gamma'')\sigma}{(\Gamma', e, \Gamma')}$$

if there exist a fresh variant $l \rightarrow r \leftarrow c$ of a rewrite rule in $R$, a non-variable position $p$ in $e$, and $\sigma = mgu(e|_p, l)$.

Remarks

- The previous inference rule is also written as
  $$(\Gamma', e, \Gamma'') \rightsquigarrow_{\sigma,p,l \rightarrow r \leftarrow c} (\Gamma', e[r]_p, c, \Gamma'')\sigma$$
or simply
  $$(\Gamma', e, \Gamma'') \rightsquigarrow \sigma (\Gamma', e[r]_p, c, \Gamma'')\sigma.$$ 

- CNC is sound: If $\Gamma \rightsquigarrow^*_\sigma \top$ then $\sigma|_{vars(G)}$ is an $R$-unifier of $\Gamma$.

- We can define basic conditional narrowing, similar to basic narrowing:
  
  Main idea: no narrowing steps should take place at positions introduced by previous narrowing substitutions.
A CNC-derivation

$$\Gamma_1 \leadsto \theta_1, p_1, l_1 \rightarrow r_1 \leftarrow c_1 \cdots \leadsto \theta_{n-1}, p_{n-1}, l_{n-1} \rightarrow r_{n-1} \leftarrow c_{n-1} \Gamma_n$$

is based on a position constraint $B_1$ for $\Gamma_1$ if $p_i \in B_i(e_i)$ for $1 \leq i \leq n - 1$, where the position constraints $B_2, \ldots, B_{n-1}$ for $\Gamma_2, \ldots, \Gamma_{n-1}$ are defined inductively by

$$B_{i+1}(e) = \begin{cases} B_i(e') & \text{if } e' \in \Gamma_i \setminus \{e_i\} \\ B(B_i(e_i), p_i, r_i) & \text{if } e' = e_i[r_i]p_i \\ Pos_F(e') & \text{if } e' \in c_i \end{cases}$$

for all $1 \leq i < n - 1$ and $e = e'\theta_i \in \Gamma_{i+1}$, with $B(B_i(e_i), p_i, r_i)$ abbreviating the set of positions

$$(B_i(e_i) \setminus \{q \in B_i(e_i) \mid q \geq p_i\}) \cup \{p_i \cdot q \in Pos_F(e) \mid q \in Pos_F(r_i)\}$$
Basic conditional narrowing (2)

- The position constraint on $\Gamma$ that assigns the set of positions $Pos_\mathcal{F}(e)$ to every $e$ in $G$ is denoted by $\overline{G}$.

- A CNC derivation is basic if it is based on $\overline{G}$.
  - Basic CNC has much smaller search space than CNC.

CNC is complete for:
- semi-complete 1-CTRSs
- semi-confluent 1-CTRSs w.r.t. normalizable substitutions
- level-semi-complete 2-CTRSs
- level-complete 3-CTRSs

Basic conditional narrowing is complete for:
- decreasing and confluent 1-CTRSs
- semi-complete orthogonal 1-CTRSs

REFERENCE: [Middeldorp and Hamoen, 1994]
LCNC = lazy conditional narrowing calculus
The only change is inference rule [o]:

\[
\begin{align*}
\Gamma', f(s_1, \ldots, s_n) & \simeq t, \Gamma'' \\
\Gamma', s_1 = l_1, s_n = l_n, r = t, c, \Gamma'' \\
\text{if } l \rightarrow r \Leftarrow c \text{ is a fresh variant of a rewrite rule in } \mathcal{R}.
\end{align*}
\]

The other inference rules ([i], [d], [v], [t]) are like those of LNC.
\[ R = \{ \ 0 + y = y, s(x) + y \to s(x + y), \ \text{fib}(0) = \langle 0, s(0) \rangle, \ \text{fib}(s(x)) \to \langle z, y + z \rangle \iff \text{fib}(x) = \langle y, z \rangle \} \]

\[
\text{fib}(x) = \langle x, x \rangle \quad \Rightarrow [o] \quad x = s(x_1), \langle z_1, y_1 + z_1 \rangle = \langle x, x \rangle, \ \text{fib}(x_1) = \langle y_1, z_1 \rangle
\]

\[
\Rightarrow [d] \quad x = s(x_1), x_1 = x, y_1 + x_1 = x, \ \text{fib}(x_1) = \langle y_1, x \rangle
\]

\[
\Rightarrow [v], \{ z_1 \mapsto x \} \quad x = s(x_1), y_1 + x = x, \ \text{fib}(x_1) = \langle y_1, x \rangle
\]

\[
\Rightarrow [o] \quad x = s(x_1), y_1 + x = x, x_1 = 0, \langle 0, s(0) \rangle = \langle y_1, x \rangle
\]

\[
\Rightarrow [v], \{ x_1 \mapsto 0 \} \quad x = s(0), y_1 + x = x, \langle 0, s(0) \rangle = \langle y_1, x \rangle
\]

\[
\Rightarrow [d] \quad x = s(0), y_1 + x = x, 0 = y_1, s(0) = x
\]

\[
\Rightarrow [v], \{ y_1 \mapsto 0 \} \quad x = s(0), 0 + x = x, s(0) = x
\]

\[
\Rightarrow [v], \{ x \mapsto s(0) \} \quad 0 + s(0) = s(0), s(0) = s(0)
\]

\[
\Rightarrow [d] \quad 0 + s(0) = s(0), 0 = 0 \\Rightarrow [d] \quad 0 + s(0) = s(0)
\]

\[
\Rightarrow [o] \quad 0 = 0, s(0) = y_2, y_2 = s(0)
\]

\[
\Rightarrow [v], \{ y_1 \mapsto s(0) \} \quad 0 = 0, s(0) = s(0)
\]

\[
\Rightarrow [d] \quad 0 = 0, 0 = 0
\]

\[
\Rightarrow [d] \Rightarrow [d] \quad \square
\]
\[ \mathcal{R} = \{ \ 0 + y = y, s(x) + y \rightarrow s(x + y), \ fib(0) = \langle 0, s(0) \rangle, \ fib(s(x)) \rightarrow \langle z, y + z \rangle \Leftarrow fib(x) = \langle y, z \rangle \} \]

\[
\begin{align*}
\text{Computed substitution: } & \{ x \mapsto s(0) \}
\end{align*}
\]
Properties of LCNC

Theorem

Let $\mathcal{R}$ be a confluent 1-CTRS and $\Gamma$ a system of equations. For every normalized unifier $\theta$ of $\Gamma$ there exists an LCNC-refutation $\Gamma \Rightarrow_{\sigma}^{*} \Box$ respecting leftmost equation selection strategy such that $\sigma \leq \theta [\text{vars}(\Gamma)]$.

Theorem

Let $\mathcal{R}$ be an arbitrary CTRS and $\Gamma \Rightarrow_{\theta}^{*} \top$ be a basic CNC-refutation. For every selection function $S$ there exists an LCNC-refutation respecting $S$ such that $\sigma \leq \theta [\text{vars}(\Gamma)]$.

Theorem

Let $\mathcal{R}$ be a terminating and level-confluent CTRS. For every $\mathcal{R}$-unifier $\theta$ of a system $\Gamma$ there exists an LCNC-refutation $\Gamma \Rightarrow_{\sigma}^{*} \Box$ such that $\sigma \leq \theta [\text{vars}(\Gamma)]$. 
We will consider equations of two kinds: \( s = t \) and \( s \uparrow t \)

**Definition**

Let \( X \) be a set of variables. A system of equations \( \Gamma : e_1, \ldots, e_n \) is \( X \)-deterministic if

\[
\text{vars}(s_i) \subseteq X \cup \bigcup_{j=1}^{i-1} \text{vars}(e_j) \quad \text{when} \quad e_i \text{ is } s_i \uparrow t_i
\]

\[
\text{vars}(e_i) \subseteq X \cup \bigcup_{j=1}^{i-1} \text{vars}(e_j) \quad \text{when} \quad e_i \text{ is } s_i = t_i
\]

A CTRS \( \mathcal{R} \) is **deterministic** if it is made of rewrite rules of the form \( l \rightarrow r \leftarrow c \) where \( c \) is an \( \text{vars}(l) \)-deterministic system of equations. \( \mathcal{R} \) is **fresh** if \( \text{vars}(t) \cap \text{vars}(l) = \emptyset \) for every \( s \uparrow t \) in the condition \( c \) of any rewrite rule \( l \rightarrow r \leftarrow c \) from \( \mathcal{R} \).

When rewriting with deterministic CTRSs, \( = \) is interpreted as joinability \((\downarrow_{\mathcal{R}})\), and \( \uparrow \) as reducibility \((\rightarrow_{\mathcal{R}})\):

- If \( \mathcal{R} \) is deterministic then \( s \rightarrow_{\mathcal{R}} t \) if there exist \( l \rightarrow r \leftarrow c \in \mathcal{R} \), \( p \in Pos_F(s) \) and \( \theta \) such that \( s|_p = l\theta \), \( t = s[r\theta]|_p \), and for all \( e_i \in c \):
  - \( s_i\theta \downarrow_{\mathcal{R}} t_i\theta \) if \( e_i \) is \( s_i = t_i \); \( s_i\theta \rightarrow^*_{\mathcal{R}} t_i\theta \) if \( e_i \) is \( s_i \uparrow t_i \).
Deterministic CTRSs and $X$-deterministic goals

Example

$\mathcal{R} = \{ 0 + y \rightarrow y, \quad fst(\langle x, y \rangle) \rightarrow x, \\
s(x) + y \rightarrow s(x + y), \quad snd(\langle x, y \rangle) \rightarrow y, \\
fib(0) \rightarrow \langle 0, s(0) \rangle, \\
fib(s(x)) \rightarrow \langle y, y + z \rangle \leftarrow fib(x) \triangleright \langle y, z \rangle \}$

The goal

$\Gamma : fib(s(x)) \triangleright \langle s(s(s(0))), y \rangle, y \triangleright s(z)$

is $\{x\}$-deterministic.
Given an $X$-deterministic goal $\Gamma$ and a deterministic CTRS $R$

Compute a complete set of $X$-normalized $R$-unifiers of $\Gamma$. A substitution $\theta$ is $X$-normalized if $\theta(x)$ is an $R$-normal form for all $x \in X$.

$\text{LCNC}_{\ell}^+$: refinement of LCNC adjusted to resolve this problem

- Works on terms from $\mathcal{T}(\mathcal{F} \cup \mathcal{F}^\dagger, \mathcal{V} \cup \mathcal{V}^\dagger)$ where $\mathcal{F}^\dagger$ (resp. $\mathcal{V}^\dagger$) is the set of marked function symbols (resp. marked variables). The purpose of marking is to avoid computing many non-normalised solutions.

- We write $t^\dagger$ for the term obtained from $t$ by marking its root symbol, if not already marked.

- We write $u(t)$ for the term obtained by removing all markers from $t$. E.g., $u(f^\dagger(x, g(y^\dagger))) = f(x, g(y))$
Inference rules (1)

Same as LCNC, except that the leftmost equation is always selected, and the inference rules $[i]$, $[d]$, $[v]$, $[t]$ are adjusted as follows:

**[i]**\[\begin{align*}
  f(s_1, \ldots, s_n) \triangleright x, \Gamma & \quad \Rightarrow \quad (s_1 \triangleright x_1, \ldots, s_n \triangleright x_n, \Gamma)\sigma \\
  f(s_1, \ldots, s_n) \simeq x^\dagger, \Gamma & \quad \Rightarrow \quad (s_1 \triangleright x_1^\dagger, \ldots, s_n \triangleright x_n^\dagger, \Gamma)\sigma'
\end{align*}\]

with $(s \simeq t) \in \{s = t, t = s, s \triangleright t\}$, $\sigma = \{x \mapsto f(x_1, \ldots, x_n)\}$, $x^\dagger \mapsto f^\dagger(x_1, \ldots, x_n)$, $\sigma' = \{x, x^\dagger \mapsto f^\dagger(x_1, \ldots, x_n)\}$, and $x_1, \ldots, x_n$ fresh variables.

**[d]**\[\begin{align*}
  f(s_1, \ldots, s_n) \simeq f(t_1, \ldots, t_n), \Gamma & \quad \Rightarrow \quad s_1 \simeq t_1, \ldots, s_n \simeq t_n, \Gamma \\
  f(s_1, \ldots, s_n) \simeq f^\dagger(t_1, \ldots, t_n), \Gamma & \quad \Rightarrow \quad s_1 \simeq t_1^\dagger, \ldots, s_n \simeq t_n^\dagger, \Gamma
\end{align*}\]

with $\simeq \in \{=, \triangleright\}$

**[v]**\[\begin{align*}
  s_1 \simeq t_1, \ldots, s_n \simeq t_n, \Gamma & \quad \Rightarrow \quad s_1^\dagger \simeq t_1^\dagger, \ldots, s_n^\dagger \simeq t_n^\dagger, \Gamma \\
  f^\dagger(s_1, \ldots, s_n) \simeq f^\dagger(t_1, \ldots, t_n), \Gamma & \quad \Rightarrow \quad f^\dagger(s_1^\dagger, \ldots, s_n^\dagger) \simeq f^\dagger(t_1^\dagger, \ldots, t_n^\dagger), \Gamma
\end{align*}\]

with $\triangleright \in \{=, \triangleright\}$
The calculus \( \text{LCNC}^\dagger \)

Inference rules (2)

\[
[\forall] \quad \frac{x^\dagger \simeq s, \Gamma}{\Gamma\theta'} \quad \text{if } s \notin \mathcal{V} \cup \mathcal{V}^\dagger \quad \frac{s \simeq x^\dagger, \Gamma}{\Gamma\theta'} \quad \frac{s \triangleright x, \Gamma}{\Gamma\theta}
\]

with \( x \notin \text{vars}(u(s)) \), \( \simeq \in \{=, \triangleright\} \),

\( \theta = \{x \mapsto s, x^\dagger \mapsto s^\dagger\} \), and

\( \theta' = \{x, x^\dagger \mapsto s^\dagger\} \cup \{y \mapsto y^\dagger \mid y \in \text{vars}(u(s))\} \)

\[
[t] \quad \frac{s \simeq t, \Gamma}{\Gamma} \quad \text{if } u(s) = u(t) \text{ and } \simeq \in \{=, \triangleright\}
\]

**Notation:** \( \Gamma^\dagger \) is the result of replacing all variables \( x \) with \( x^\dagger \) in \( \Gamma \)

**Theorem**

*Let \( \mathcal{R} \) be a deterministic CTRS and \( \theta \) a normalized \( \mathcal{R} \)-unifier of \( \Gamma \). There exists an \( \text{LCNC}^\dagger \ell \)-refutation \( G^\dagger \Rightarrow^* \square \) such that

\( u(\sigma) \leq \theta [\text{vars}(\Gamma)] \)
### The calculus LCNC

Nondeterminism due to selection of inference rules

#### For unoriented equation $s = t$

<table>
<thead>
<tr>
<th>$\text{root}(s) \backslash \text{root}(t)$</th>
<th>$\mathcal{F}^+$</th>
<th>$\mathcal{F}_C$</th>
<th>$\mathcal{F}_D$</th>
<th>$\mathcal{V}^+$</th>
<th>$\mathcal{V}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{F}^+$</td>
<td>$[t]; [d]$</td>
<td>$[t]; [d]$</td>
<td>$[t]; [d], [o]_2$</td>
<td>$[v]$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$\mathcal{F}_C$</td>
<td>$[t]; [d]$</td>
<td>$[t]; [d]$</td>
<td>$[o]_2$</td>
<td>$[i], [v]$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$\mathcal{F}_D$</td>
<td>$[t]; [d], [o]_1$</td>
<td>$[o]_1$</td>
<td>$[t]; [d], [o]_1, [o]_2$</td>
<td>$[o]_1, [i], [v]$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$\mathcal{V}^+$</td>
<td>$[v]$</td>
<td>$[i], [v]$</td>
<td>$[o]_2, [i], [v]$</td>
<td>$[t]; [v]$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$\mathcal{V}$</td>
<td>$\times$</td>
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<td>$\times$</td>
<td>$\times$</td>
</tr>
</tbody>
</table>

#### For oriented equation $s \triangleright t$

<table>
<thead>
<tr>
<th>$\text{root}(s) \backslash \text{root}(t)$</th>
<th>$\mathcal{F}^+$</th>
<th>$\mathcal{F}_C$</th>
<th>$\mathcal{F}_D$</th>
<th>$\mathcal{V} \cup \mathcal{V}^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{F}^+$</td>
<td>$[t]; [d]$</td>
<td>$[t]; [d]$</td>
<td>$[t]; [d]$</td>
<td>$[v]$</td>
</tr>
<tr>
<td>$\mathcal{F}_C$</td>
<td>$[t]; [d]$</td>
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<td>$\times$</td>
<td>$[i], [v]$</td>
</tr>
<tr>
<td>$\mathcal{F}_D$</td>
<td>$[t]; [d], [o]_1$</td>
<td>$[o]_1$</td>
<td>$[t]; [d], [o]_1$</td>
<td>$[o]_1, [i], [v]$</td>
</tr>
<tr>
<td>$\mathcal{V}^+$</td>
<td>$[v]$</td>
<td>$[v]$</td>
<td>$[v]$</td>
<td>$[t]; [v]$</td>
</tr>
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</tbody>
</table>
**Main idea**: Like LNC, the calculus LCNC can apply eagerly variable elimination for descendants of parameter-passing equations without losing completeness.

- Descendants of parameter-passing equations are defined in exactly the same way as for LNC; we write $s \triangleright t$ to distinguish them from other kinds of equations.
- $\text{LCNC}_{\ell}^{\text{eve}}$: adjustment of $\text{LCNC}_{\ell}^{\dagger}$ with the following strategy to solve parameter-passing equations:

<table>
<thead>
<tr>
<th>root(s)</th>
<th>root(t)</th>
<th>$\mathcal{F}_C$</th>
<th>$\mathcal{F}_D$</th>
<th>$\mathcal{V}$</th>
<th>$\mathcal{F}_C^{\dagger}$</th>
<th>$\mathcal{V}_D^{\dagger}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{F}_C^{\dagger}$</td>
<td>[t]; [d]</td>
<td>[t]; [d]</td>
<td>[v]</td>
<td>×</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathcal{F}_C$</td>
<td>[t]; [d]</td>
<td>×</td>
<td>[v]</td>
<td>×</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathcal{F}_D$</td>
<td>[o]₁</td>
<td>[t]; [d], [o]₁</td>
<td>[v]</td>
<td>×</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathcal{V}_C^{\dagger}$</td>
<td>[v]</td>
<td>[v]</td>
<td>[v]</td>
<td>×</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathcal{V}$</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
BAD NEWS: \( \text{LCNC}_\ell^{\text{eve}} \) is incomplete for left-linear deterministic CTRSs.

Example

\[ \mathcal{R} = \{ f(x) \rightarrow x, g(x, y) \rightarrow x \leftarrow x \triangleright y \} \]

\[ \Gamma : g(x, f(y)) = a \]

\( \mathcal{R} \) is left-linear and deterministic, but not fresh; \( \theta = \{ x \mapsto a, y \mapsto a \} \) is a normalized solution of \( \Gamma \). \( \theta \) can not be computed with \( \text{LCNC}_\ell^{\text{eve}} \), because the only maximal \( \text{LCNC}_\ell^{\text{eve}} \)-derivation is

\[
\begin{align*}
\Gamma^\dagger & \Rightarrow [\varnothing] \\
& \Rightarrow [\nu], \{ x_1 \mapsto x^\dagger, x_i^\dagger \mapsto x^\dagger \} \\
& \Rightarrow [\nu], \{ y_1 \mapsto f(y^\dagger), y_i^\dagger \mapsto f(y^\dagger) \} \\
& \Rightarrow [\nu], \{ x \mapsto f(y^\dagger), x^\dagger \mapsto f(y^\dagger) \}
\end{align*}
\]

\( x^\dagger \triangleright x_1, f(y^\dagger) \triangleright y_1, x_1 \triangleright y_1, x_1 = a \\
f(y^\dagger) \triangleright y_1, x^\dagger \triangleright y_1, x^\dagger = a \\
x^\dagger \triangleright f(y^\dagger), x^\dagger = a \\
f^\dagger(y^\dagger) = a \)
**BAD NEWS:** \( \text{LCNC}^\text{eve}_\ell \) is incomplete for left-linear deterministic CTRSs.

**Example**

\( \mathcal{R} = \{f(x) \rightarrow x, g(x, y) \rightarrow x \Leftarrow x \triangleright y\} \)

\( \Gamma : g(x, f(y)) = a \)

\( \mathcal{R} \) is left-linear and deterministic, but not fresh; \( \theta = \{x \mapsto a, y \mapsto a\} \) is a normalized solution of \( \Gamma \). \( \theta \) can not be computed with \( \text{LCNC}^\text{eve}_\ell \), because the only maximal \( \text{LCNC}^\text{eve}_\ell \)-derivation is

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\end{align*}
\]

**GOOD NEWS:** \( \text{LCNC}^\text{eve}_\ell \) is complete for left-linear fresh deterministic CTRSs
**Definition**

A substitution $\theta$ is a **strict solution** of an equation

- $s \triangleright t$ if $s\theta \rightarrow^* _{\mathcal{R}} t\theta$ and $t\theta \in \mathcal{T}(\mathcal{F}_c, \mathcal{V})$ (that is, $t\theta$ is a term without defined function symbols)

- $s = t$ if there exists a term $u \in \mathcal{T}(\mathcal{F}_c, \mathcal{V})$ such that $s\theta \rightarrow^* _{\mathcal{R}} u$ and $t\theta \rightarrow^* _{\mathcal{R}} u$.

$\theta$ is a **strict solution** of a goal $\Gamma$ if it is a strict solution of every equation from $\Gamma$.

**Challenge**: Find a refinement of LCNC which computes a complete set of strict solutions of a given goal $\Gamma$. 
Unification of strict equations for deterministic CTRSs

Definition

A substitution $\theta$ is a strict solution of an equation

- $s \triangleright t$ if $s\theta \rightarrow^*_R t\theta$ and $t\theta \in T(F_c, V)$ (that is, $t\theta$ is a term without defined function symbols)
- $s = t$ if there exists a term $u \in T(F_c, V)$ such that $s\theta \rightarrow^*_R u$ and $t\theta \rightarrow^*_R u$.

$\theta$ is a strict solution of a goal $\Gamma$ if it is a strict solution of every equation from $\Gamma$.

Challenge: Find a refinement of LCNC which computes a complete set of strict solutions of a given goal $\Gamma$.

- The calculus $\text{LCNC}^S_\ell$ [Marin and Middeldorp, 2004] was designed for this purpose.
The lazy narrowing calculus \( \text{LCNC}_s^\ell \)

Inference rules (1)

\[ [0] \quad \frac{f(s_1, \ldots, s_n) \simeq t, \Gamma}{s_1 \triangleright l_1, \ldots, s_n \triangleright l_n, c, \Gamma} \text{ where } \simeq \in \{ =, =^{-1}, \triangleright, \triangleright_\prime \}
\]

if \( f(l_1, \ldots, l_n) \rightarrow r \Leftarrow c \) is a fresh variant of a rewrite rule in \( \mathcal{R} \)

\[ [i] \quad \frac{g(s_1, \ldots, s_n) \simeq x, \Gamma}{(s_1 \simeq x_1, \ldots, s_n \simeq x_n, \Gamma)\theta} \text{ where } \simeq \in \{ =, =^{-1}, \triangleright \}, g \in \mathcal{F}_c
\]

if \( g(s_1, \ldots, s_n) \not\in \mathcal{T}(\mathcal{F}_c, \mathcal{V}), \theta = \{ x \mapsto g(x_1, \ldots, x_n) \}. \)

\[ g(s_1, \ldots, s_n) \triangleright x, \Gamma
\]

\[ (s_1 \triangleright x_1, \ldots, s_n \triangleright x_n, \Gamma)\theta' \text{ if } \theta' = \{ x \mapsto f(x_1, \ldots, x_n) \}\]

\[ [d] \quad \frac{g(s_1, \ldots, s_n) \simeq g(t_1, \ldots, t_n), \Gamma}{s_1 \simeq t_1, s_n \simeq t_n, \Gamma} \text{ where } \simeq \in \{ =, \triangleright \} \text{ and } g \in \mathcal{F}_c
\]

\[ f(s_1, \ldots, s_n) \simeq f(t_1, \ldots, t_n), \Gamma
\]

\[ s_1 \triangleright t_1, \ldots, s_n \triangleright t_n, \Gamma
\]
The lazy narrowing calculus $\text{LCNC}^s_\ell$

Inference rules (2)

\[\text{[\(\triangleright\)] } \frac{x \triangleright s, \Gamma}{\Gamma\theta} \quad \text{were } s \not\in \mathcal{V} \]

\[\text{[\(\triangleright\)] } \frac{s \triangleright x, \Gamma}{\Gamma\theta} \quad \text{where } s \in \mathcal{T}(\mathcal{F}_c, \mathcal{V}) \]

\[\text{[\(\triangleright\)] } \frac{x \simeq s, \Gamma}{\Gamma\theta} \quad \text{where } s \in \mathcal{T}(\mathcal{F}_c, \mathcal{V}) \setminus \mathcal{V} \]

\[\text{where } x \not\in \text{vars}(s), \simeq \in \{=, \triangleright\}, \text{and } \theta = \{x \mapsto s\} \]

\[\text{[\(\triangleright\)] } \frac{s \triangleright s, \Gamma}{\Gamma} \quad \frac{s \simeq s, \Gamma}{\Gamma} \quad \text{where } \simeq \in \{=, \triangleright\} \text{ and } s \in \mathcal{T}(\mathcal{F}_c, \mathcal{V}) \]
Theorem

Let $\mathcal{R}$ be a deterministic CTRS and $\theta$ an $\mathcal{R}$-normalized strict solution of $\Gamma$. Then there exists an $\text{LCNC}^S_\ell$-refutation $\Gamma \Rightarrow^*_\sigma \square$ such that $\sigma \leq \theta [\text{vars}(\Gamma)]$. 

Marin

Unification by Narrowing
Higher-order extensions of the narrowing calculus

- Functional programming operates with functions as values
- The lambda calculus is suitable to express functional computations (function abstractions and function calls)
  ⇒ it is natural to try to extend narrowing to solve systems of equations between λ-terms

\[ t ::= x \quad \text{variable} \]
\[ \lambda x.t \quad \text{abstraction} \]
\[ (t \, t) \quad \text{application} \]

where \( x \) ranges over a countably infinite set of variables.

### Conversion rules for λ-terms

λ-terms are identified modulo the following conversion rules:

- \( \lambda x.t \rightarrow \lambda y.(t\{x \mapsto y\}) \) if \( y \in V \setminus \text{vars}(t) \) (\( \alpha \)-conversion)
- \( (\lambda x.s) \, t = s\,\{x \mapsto t\} \) (\( \beta \)-conversion)
- \( (\lambda x.(t \, x) = t \) if \( x \in V \setminus \text{vars}(t) \) (\( \eta \)-conversion)
Rewriting systems for \( \lambda \)-terms

Usually, higher-order \( E \)-unification is performed between simply-typed \( \lambda \)-terms, which can be represented in a standard form called long \( \beta \eta \)-normal form.

TRSs have been generalised to pattern rewrite systems (PRS), and CTRSs to conditional PRSs.

Narrowing has been generalised to higher-order lazy narrowing with PRSs.

- 1998: Prehofer proposed lazy narrowing calculus for PRS \( \mathcal{R} \), called LN. LN performs higher-order \( \mathcal{R} \)-preunification.
- Main challenge: reduce the search space for solutions

- since 2000: several refinements of LN which reduce nondeterminism have been proposed.