4 Risch's Algorithm

Recall:

<u>Thm 5</u> (Liouville) Let K be a differential field with algebraically closed constant field C, and let $f \in K$. Then f is elementary integrable iff and only iff

$$f = D(g) + \sum_{i=1}^{n} \gamma_i \frac{D(v_i)}{v_i}$$

for some $g \in K$, $\gamma_1 \ldots \gamma_n \in C$, $v_1 \ldots v_n \in K$.

<u>Goal:</u> an algorithm for deciding whether $f \in K$ can be written in this form, for the case when $K = C(x_1, x_2 \ldots x_n)$ with $D(x_1) = 1$ and each other x_i is transcendental exponential or transcendental logarithmic over $C(x_1 \ldots x_{i-1})$ and const $K = C$.

Idea: write $K = k(x)$ for $x = x_n$ and $k = C(x_1 \dots x_{n-1})$ and use suitable generalizations of the algorithms for $(C(x), \frac{d}{dx})$.

Recall: For $K = C(x)$ and $D = \frac{d}{dx}$ and $f \in K$, we proceed as follows:

$$f = p + \frac{a}{d}$$ with $\deg a < \deg d$ (polynomial division)

$p \downarrow$ easy to integrate

$\frac{a}{d} \downarrow$ Hermite Reduction

$$= D(g) + \frac{u}{v}$$ with $\deg u < \deg v$ and v monic sqf.

$D(g) \downarrow$ easy to integrate

$\frac{u}{v} \downarrow$ Rothstein-Trager

$$= \sum_{i=1}^{n} \gamma_i \frac{D(v_i)}{v_i}$$

$\downarrow$ easy to integrate.

<u>Plan for today:</u> Generalize Hermite Reduction and Rothstein-Trager to $k(x)$.

<u>Plan for next week:</u> Generalize polynomial integration to $k(x)$.

(4a) Hermite Reduction

In $C(x)$ with $D = \frac{d}{dx}$, this works because $\gcd(p, \frac{d}{dx} p) = 1$ for square-free polynomials $p \in C[x]$. This generalizes as follows.

<u>Lemma</u> Let $K = k(x)$ with $\mathrm{const}\, K = \mathrm{const}\, k$, $p \in k[x]$ monic sqf with $\deg p > 0$.

(1) If x is logarithmic over k then $\gcd(p, Dp) = 1$

(2) If x is exponential over k then $\gcd(p, Dp) = 1 \iff x \nmid p$.

Proof.

(1) First assume p is irreducible. We have

$$D p = \underbrace{\delta_D(p)}_{\substack{\deg < \deg p \\ \text{because} \\ lc(p) = 1 \in \operatorname{const} K}} + \underbrace{\frac{d}{dx}(p)}_{\deg < \deg p} \cdot \underbrace{D(x)}_{\substack{\in K \setminus \{0\} \text{ because} \\ x \text{ is logarithmic} \\ \text{and } \operatorname{const} K = \operatorname{const} k}}$$

Therefore $\deg(Dp) < \deg p$. Therefore $\gcd(p, Dp) = 1$.

For the general case, write $p = p_1 \cdots p_n$ for the factorization of p into irreducibles. Then $\gcd(p_i, p_j) = 1$ $(i \neq j)$ because p is sqf. Now

$$\gcd(p, Dp) = \gcd\Big(\prod_i p_i,\ \sum_i (Dp_i) \prod_{j \neq i} p_j\Big)$$

$$\underset{\substack{\uparrow \\ \text{easy}}}{=} \prod_i \gcd(p_i, Dp_i) = \prod_i 1 = 1.$$

(2) If $x \mid p$, say $p = xq$ for some $q \in k[x]$ then $Dp = \underbrace{D(x)}_{= ux} \cdot q + x\, D(q) = x \cdot (uq + D(q))$,

so $x \mid Dp$ as well and $x \mid \gcd(p, Dp)$ implies $\gcd(p, Dp) \neq 1$.

If p is irred and $p \neq x$ and $\gcd(p, Dp) \neq 1$ then $p = \gcd(p, Dp)$, so

$$p = \frac{D(p)}{lc(Dp)} = \frac{D(p)}{du} \quad \text{where } d = \deg p.$$

$$\Rightarrow \quad \frac{D(p)}{p} = du.$$

But then $\frac{p}{x^u} \in k(x) \setminus k$ is a constant.

This is impossible.

If p is not irreducible, argue as in part (1) ☒

The exceptional case $x \mid p$ in part (2) is not a limitation if we split the given integrand $f \in k(x)$ as

$f = p + \frac{a}{d}$ with $p \in k[x, x^{-1}]$ and $a, d \in k[x]$ with $\deg a < \deg d$ and $x \nmid d$.

In other words, we postpone the handling of this subtlety to the polynomial integrator.

Alg 2 (Hermite Reduction)

Input: $f = \frac{a}{b} \in k(x)$ with $\deg a < \deg d$

x logarithmic over k or

x exponential over k and $x \nmid d$.

Output: $g, h \in k(x)$ with $f = D(g) + h$

st h has a square free

denominator.

[instructions are literally as in Alg 1

steps (2) - (7) with "$g = 0$" in step 3

using D instead of $\frac{d}{dx}$ as derivative].

(4b) Rothstein - Trager

Task: given $f = \frac{p}{q} \in k(x)$ with $\deg p < \deg q$

and q monic sqf and $x \nmid q$ if x is

exponential over k, find, if possible,

a representation

$$f = \sum_{i=1}^{n} \gamma_i \frac{D(v_i)}{v_i}$$

with $\gamma_1 \ldots \gamma_n \in \overline{\mathrm{Const}\, k}$ and

$v_1 \ldots v_n \in k(\gamma_1 \ldots \gamma_n)(x)$.

If such a representation exists at all, we may assume wlog that the v_i are monic squarefree and pw coprime, and that the γ_i are pw distinct (similar argument as on page 24)

Then $\gcd(v_i, D(v_i)) = 1$ for all i

and $q = v_1 \dots v_n$

and $p = \sum_{i=1}^{n} \gamma_i D(v_i) \prod_{j \neq i} v_j$

and $Dq = \sum_{i=1}^{n} D(v_i) \cdot \prod_{j \neq i} v_j$.

<u>Claim:</u> $\gcd(q, p - \gamma_i Dq) = v_i$ $(i = 1..n)$.

rhs | lhs clear

lhs | rhs : if $u \mid$ lhs then $u \mid q$ and then $u \mid v_j$ for some j. If $j = i$, we are done. Otherwise

$$v_j \mid p - \gamma_i Dq = \sum_{k=0}^{n} (\gamma_k - \gamma_i) D(v_k) \prod_{l \neq k} v_l$$

$\Leftrightarrow$ $v_j \mid (\gamma_j - \gamma_i) D(v_j)$

$\Leftrightarrow$ $v_j \mid D(v_j)$ or $\gamma_i = \gamma_j$.

Both are impossible.

By a similar argument, $\gcd(q, p - \gamma Dq) = 1$ for any $\gamma \in \bar{C} \setminus \{\gamma_1 \dots \gamma_n\}$.

Consequently, all γ_i must be among the constant (!) roots of $\mathrm{res}_x(q, p - zDq) \in k[z]$.

<u>Alg 3</u> (Rothstein-Trager)

Input: $f = \frac{p}{q} \in k(x)$ with $\deg p < \deg q$, q monic sqf.

x logarithmic over k or
x exponential over k and $x \nmid q$.

Output: $\gamma_1 \dots \gamma_n \in \bar{C}$ and $u, v_1 \dots v_n \in k(\gamma_1 \dots \gamma_n)[x]$ st $f = u + \sum_{i=1}^{n} \gamma_i \frac{D(v_i)}{v_i}$ or $\perp$ if no such data exists.

(1) $R := \mathrm{res}_x(q, p - z\,D(q)) \in k[z]$.

(2) Let $\gamma_1 \dots \gamma_n \in \bar{C}$ be the roots of those factors of R which belong to $C[z]$.

(3) for each i, set $v_i := \gcd(q, p - \gamma_i D(q))$

(4) $u := f - \sum_{i=1}^{n} \gamma_i \frac{D(v_i)}{v_i}$.

(5) If $u \in k[x]$ return $\gamma_1 \dots \gamma_n, u, v_1 \dots v_n$ else return $\perp$.

Ex:

(1) $f = \dfrac{1}{e^x+1} \in \mathbb{Q}(x, e^x)$

$\mathrm{res}_y(y+1,\ 1-zy) = 1+z = 0 \Leftrightarrow z = \underline{\underline{-1 = \gamma_1}}$

$\gcd(y+1,\ 1-(-y)) = \underline{\underline{y+1 = v_1}}$

$$\sum_i \gamma_i \frac{D(v_i)}{v_i} = -1 \cdot \frac{D(e^x+1)}{e^x+1} = -\frac{e^x}{e^x+1}$$

$u = f - \left(-\dfrac{e^x}{e^x+1}\right) = \dfrac{1+e^x}{1+e^x} = 1$ remains to be integrated.

(2) $f = \dfrac{1}{1+\log x} \in \mathbb{Q}(x, \log x)$

$\mathrm{res}_y\left(y+1,\ 1 - z\frac{1}{x}\right) = \frac{1}{x}(x-z) = 0 \Leftrightarrow z = x \notin \overline{C}$

$\Rightarrow\ u = f - 0 = f \notin \mathbb{Q}(x)[\log x]$

$\Rightarrow\ f$ is not elementary integrable.