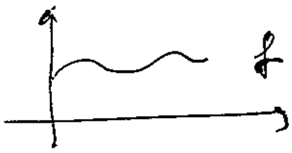
1. Introduction

Computer Analysis

$$\text{finite} \not\longleftrightarrow \text{infinite}$$

Analysis: the theory of real (or complex) functions.



Recall: A function $f: A \to B$ is a subset $f \subseteq A \times B$ with $\forall x \in A\ \exists!\, y \in B : (x,y) \in f$ (written then "$f(x) = y$")

Want: algorithms for computing with functions.

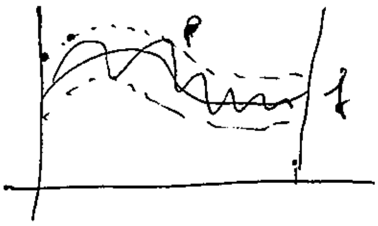
Problem: when $A, B \subseteq \mathbb{R}$, there is no way of storing a function $f: A \to B$ on a (finite) computer: too much data!

Not even a single number $x \in \mathbb{R}$ can be stored in general on a computer, because $\mathbb{R}$ is uncountable and there are only countably many bit strings.

<u>Hence:</u> There do not exist algorithms which take as input "a realvalued function" and do something with it.

<u>Usual workaround:</u> Approximation.

<u>Ex:</u> Represent $f: [0,1] \to \mathbb{R}$ by a polynomial $p \in \mathbb{Q}[x]$ with $\max_{t \in [0,1]} |f(t) - p(t)| < \varepsilon$ for some fixed $\varepsilon > 0$, and compute with p instead of f. If done well, we can hope that the result for p will be a good approximation to the (hypothetical) result for f.



This is fair enough for almost every application, but has nothing to do with the topic of this course.
(cf "numerics" lectures)

<u>Note:</u> We cannot hope to get an "exact" answer if already the input is just an approximation. On the other hand, computer algebra systems can find "exact" formulas like

$$\underbrace{\int \frac{1}{1+e^x}\,dx}_{\text{input}} = \underbrace{x - \log(1+e^x)}_{\text{output}}$$

How is this possible?

Topic of this course: Algorithms for answering questions from analysis (differentiation, integration, solving ODEs, etc) in a formal (= non-approximate = algebraic) fashion.

The underlying algebraic theory for this is <u>differential algebra</u>. Basic idea: consider domains with three operations $+, \cdot, D$. (usual algebra: just $+$ and $\cdot$)

<u>Def 1</u> Let R be a commutative ring / a field, and let $D: R \to R$ be a map with

$$D(a+b) = D(a) + D(b) \qquad (a, b \in R)$$

$$D(a \cdot b) = D(a)\, b + a\, D(b) \qquad (a, b \in R)$$

Then D is called a <u>derivation</u> on R, and the pair (R, D) is called a <u>differential ring</u> / <u>differential field</u>.

An element $c \in R$ is called a <u>constant</u> wrt D if $D(c) = 0$. We write

$$\operatorname{const}(R) := \operatorname{const}_D(R) := \{ c \in R \mid D(c) = 0 \}$$

for the set of all constants.

Global assumption: $\mathbb{Q} \subseteq R$ for all rings in this course

<u>Ex:</u>

(1) $R = C^\infty([0,1], \mathbb{R})$ with $D: R \to R$, $D(f) = f'$ is a differential ring. $\operatorname{const}_D(R)$ ~~contains~~ consists of all the constant functions.

(2) Let R be any ring and $D : R \to R$
$D(a) = 0$. Then (R, D) is a differential ring and $\mathrm{const}(R) = R$.

(3) Let R be any ring and $P := R[x]$ be the ring of univariate polynomials in x with coefficients in R, and

$$D : P \to P, \quad D(a_0 + a_1 x + \dots + a_n x^n) := a_1 + 2a_2 x + \dots + n\, a_n x^{n-1}.$$

Then (P, D) is a differential ring and $\mathrm{const}(P) = R$.
This derivation is called the <u>standard derivation</u> for $R[x]$, we write it $\frac{d}{dx} := D$.

(4) Let (R, D) be any differential ring and $P := R[x]$ and $\delta_D : P \to P$ with

$$\delta_D(a_0 + a_1 x + \dots + a_n x^n) := D(a_0) + D(a_1)x + \dots + D(a_n).$$

Then (P, δ_D) is a differential ring and $\mathrm{const}_{\delta_D}(P) = \mathrm{const}_D(R)[x]$. This derivation is called the <u>coefficient derivation</u> for $R[x]$

(5) Let $R = \mathbb{Q}[x]$ and $D: R \to R$ with

$$D(a_0 + a_1 x + \dots + a_n x^n) := (a_1 + 2a_2 x + \dots + n a_n x^{n-1}) \cdot x$$
$$= x \cdot \frac{d}{dx}(a_0 + \dots + a_n x^n).$$

Then D is a derivation on R and $\mathrm{const}_D(R) = \mathbb{Q}$.

In a sense, the element x in the differential ring $(\mathbb{Q}[x], D)$ <u>models</u> the function e^t in that $(e^t)' = e^t$ and, more generally, $p(e^t)' = p'(e^t) \cdot e^t$.

(6) If $D(p) = (1+p^2)\frac{d}{dx}(p)$ for $p \in \mathbb{Q}[x]$, then D is also a derivation. In this case, x models the function $\tan(t)$, because $\tan'(t) = 1 + \tan(t)^2$.

<u>Note:</u> Except in Ex (1), these rings do not contain "functions", and the differentiation is not defined by a limit. The letter "x" does not stand for a "real variable" but constitutes a purely algebraic object of its own.

<u>Thm 1</u> Let (R, D) be a differential ring.
Then:

(1) $D(1) = D(0) = 0$

(2) If $n \in \mathbb{N}$, $a \in R$, then $D(a^n) = n\, a^{n-1} D(a)$.

(3) If $a \in R$ has a multiplicative inverse a^{-1} (i.e. $a \cdot a^{-1} = 1$) then $D(a^{-1}) = -\dfrac{D(a)}{a^2}$. (and $D(a^n) = n\, a^{n-1} D(a)$ for all $n \in \mathbb{Z}$)

(4) $\mathrm{const}_D R$ is a subring of R (a subfield of R is a field).

<u>Proof</u>
eg for (2) $n = 0$ is (1), $n = 1$ is evident.
Assume it is true for n. Then

$$D(a^{n+1}) = D(a^n a) = D(a^n)a + a^n D(a)$$
$$= n\, a^{n-1} D(a) a + a^n D(a)$$
$$= (n+1) a^n D(a). \quad \square$$

More generally, we have for $a_1, \dots, a_n \in R$

$$D(a_1 \cdots a_n) = \sum_{k=0}^{n} D(a_k) \cdot \prod_{i \neq k} a_i$$
$$= \sum_{k=0}^{n} \frac{D(a_k)}{a_k} \cdot \prod_{i=0}^{n} a_i.$$

It also follows from Thm 1 that when $\mathbb{Q} \subseteq K$ for some differential field (K, D), then $\mathbb{Q} \subseteq \mathrm{const}_D(K)$.

Moreover, the theorem implies that in order to turn $K(x)$ into a differential field, it is sufficient to define D for x and the elements of K, because then for arbitrary

$$p = p_0 + p_1 x + \dots + p_n x^n \in K[x]$$

it necessarily follows that

$$\begin{aligned} D(p) &= D(p_0) + D(p_1 x) + \dots + D(p_n x^n) \\ &= D(p_0) + D(p_1)x + p_1 D(x) \\ &\quad + D(p_2)x + 2p_1 x D(x) \\ &\quad + \dots \\ &\quad + D(p_n)x + n p_n x^{n-1} D(x) \\ &= \delta_D(p) + \frac{d}{dx}(p) \cdot D(x) \end{aligned}$$

and for $\frac{p}{q} \in K(x)$ it necessarily follows

$$\begin{aligned} D\left(\frac{p}{q}\right) &= D\left(p \cdot \frac{1}{q}\right) = D(p)\frac{1}{q} + p\,D\left(\frac{1}{q}\right) \\ &= \frac{D(p)}{q} - \frac{p\,D(q)}{q^2} = \frac{D(p)q - p\,D(q)}{q^2} \end{aligned}$$

On the other hand, we are completly free in the choice of $D(x)$: If (K, D) already is a differential field ~~and~~, then for any arbitrary $q \in K(x)$, the definition

$$\tilde{D} : K(x) \to K(x), \quad \begin{array}{ll} \tilde{D}(a) := D(a) & (a \in K) \\ \tilde{D}(x) := q & \end{array}$$

yields a differential field $(K(x), \tilde{D})$.

This leads to the concept of differential field extensions.

__Def 2.__ Let (K, D_K), (E, D_E) be two differential fields with $\underset{\text{subfield}}{K \subseteq E}$ and $D_K(a) = D_E(a)$ for all $a \in K$. Then (E, D_E) is called a (differential field) extension of (K, D_K).

<u>Def 3:</u>
Let (E, D) be a differential field extension of (K, D_K). An element $a \in E$ is called

(a) logarithmic over K
if $\exists\, b \in K:\ D(a) = \frac{D(b)}{b}$ ("$a = \log b$")

(b) exponential over K
if $\exists\, b \in K:\ \frac{D(a)}{a} = D(b)$ ("$a = \exp b$")

(c) algebraic over K
if $\exists\, p \in K[x] \setminus \{0\}:\ p(a) = 0$

(d) primitive over K
if $\exists\, b \in K:\ D(a) = b$ ("$a = \int b$")

(e) hyperexponential over K
if $\exists\, b \in K:\ \frac{D(a)}{a} = b$ ("$a = \exp \int b$")

(f) elementary over K
if it is logarithmic, exponential or algebraic

(g) Liouvillean over K
if it is primitive, hyperexponential or algebraic.

E is called elementary / Liouvillean over K iff $\exists\, a_1 \dots a_n \in E$ such that $E = K(a_1 \dots a_n)$ and each a_i is elementary / Liouvillean over $K(a_1 \dots a_{i-1})$.

If this is the case and $K = \text{const}\, E$, we say that E is an elementary / Liouvillean field.

<u>Ex:</u>

(1) $K = \mathbb{Q}(x_1, x_2, x_3)$ together with

$$D(x_1) = 1 \qquad (x_1 \triangleq t)$$

$$D(x_2) = -2x_1x_2 \qquad (x_2 \triangleq e^{-t^2})$$

$$D(x_3) = \frac{1-2x_1x_2}{x_1+x_2} \qquad (x_3 \triangleq \log(t+e^{-t^2}))$$

is a Liouvillean field.

(2) $K = \mathbb{Q}(x)$ together with

$$D(x) = \frac{2x^2+1}{3x-7}$$

is not.