**Task 1** Evaluate the integral $\int \frac{3\mathrm{e}^{2t} - 2t\mathrm{e}^t + 3t - 18\mathrm{e}^t + 20}{(t - \mathrm{e}^t + 6)^2} dt$ using the Risch algorithm.

**Task 2** Consider the differential field $K = \mathbb{Q}(x, y)$ with $D(x) = 1$ and $D(y) = \frac{1}{x}$. Prove that $\mathrm{Const}(K) = \mathbb{Q}$.

**Task 3** Assume that $C$ is algebraically closed (if you wish), and consider the differential field $K = C(x)$ with $D(x) = x$ (so that " $x = \mathrm{e}^t$ "). Design an algorithm which for given $u, v \in K$ finds all $y \in K$ such that $D^2(y) + uD(y) + vy = 0$. More specifically:

a) Design an algorithm which finds $a, b \in \mathbb{Z}$ such that all solutions $y \in C[x, x^{-1}]$ have the form $y = \sum_{i=a}^{b} c_i x^i$ for some $c_i \in C$.

b) Design an algorithm which finds $d \in C[x]$ such that for all solutions $y = p/q$ with $p \in C[x, x^{-1}]$ and $q \in C[x]$ with $x \nmid q$ we have $q \mid d$.

**Task 4** In a computer algebra system of your choice, write a program which takes as input a linear differential equation with polynomial coefficients, $a_r(x)y^{(r)}(x) + \cdots + a_0(x)y(x) = 0$, and returns as output a recurrence equation $b_0(n)y_n + b_1(n)y_{n-1} + \cdots + b_s(n)y_{n-s} = 0$ for the coefficient sequence $(y_n)$ of any series solution $y(x) = \sum_n y_n x^n$ of the input differential equation.

**Task 5** What are the groups that can appear as differential Galois group of a first order scalar equation $y' = ay$, where $a$ is a fixed element of some differential field $K$?

**Task 6** It was explained in the lecture that every scalar equation $a_r y^{(r)} + \cdots + a_0 y = 0$ can be rephrased as a matrix equation $Y' = AY$ for a suitably chosen matrix $A \in K^{r \times r}$. It was mentioned, though not explained in detail, that there is also some sort of converse. Work this out for the case $r = 2$. More precisely: Suppose that $(y_1, y_2)$ is a solution of the first order matrix equation

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

where $a_{11}, a_{12}, a_{21}, a_{22}$ are known elements of $K$, and show that both $y_1$ and $y_2$ are also solutions of some scalar equation with coefficients in $K$ (possibly of higher order).

**Task 7**

a) Construct a linear differential equation with polynomial coefficients which has $\mathrm{e}^{x^2}$ and $\frac{x-1}{x^2+1}$ among its solutions.

b) Show that there does not exist a linear differential equation with polynomial coefficients which has $\mathrm{e}^{\mathrm{e}^x}$ among its solutions. (You may use without proof that $\mathrm{e}^x$ is not algebraic.)

The use of computer algebra for polynomial arithmetic and linear algebra subtasks is encouraged, but no built-in functions or add-on packages for handling integrals or differential equations are allowed. Please document your calculations accordingly.