Rewriting in Computer Science and Logic (326.065, SS 2013) Exercises, Part 1

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Total points: 100.

Part 1. Abstract Reduction

1. (5 points) Find a proof or bring a counterexample:

   (a) The reflexive closure of the transitive closure is the same as the transitive closure of the reflexive closure.

   (b) $(\overset{+}{\leftrightarrow}) = (\overset{+}{\rightarrow}) \cup (\overset{+}{\rightarrow})^{-1}$.

2. (5 points) Give an example of ARS $(A, \rightarrow)$ such that there exists an element $a \in A$ with the following properties:

   - a has a normal form.
   - a has at most one normal form.
   - There exist reductions starting from a which do not converge.

3. (5 points) Give a counterexample to the proposition obtained from Newman's Lemma by dropping the requirement on $\rightarrow$ to be terminating.

Part 2. Syntax and Semantics

1. (4 points) Let $\mathcal{F}$ be the signature consisting of the nullary symbol 0, the unary symbol s, and the binary symbols add and $mult$. Consider the term t:

$$\begin{aligned} t = add(&mult(add(s(0), s(mult(s(s(0)), s(s(s(0)))))),\\ &\qquad s(s(add(s(0), s(s(0)))))),\\ &add(0, s(0))) \end{aligned}$$

   (a) Picture the term as a tree.

   (b) What is $\mathcal{P}os(t)$?

(c) What is $|t|$ (the size of t)?

(d) Give all subterms of t.

(e) What are the positions of function symbols?

(f) Which terms are denoted by the following expressions?

i. $t|_{112}$

ii. $t|_{221}$

iii. $t[s(0)]_{111}$

iv. $t[s(0)]_{\epsilon}$

v. $t[add(s(0), 0)]_{11}$

2. (4 points, Ex. 3.1 from the book, [BN3.1]) Let $\mathcal{F}$ be the signature consisting of the binary function symbol f. Let t be a term from $T(\mathcal{F}, \mathcal{V})$. What are the possible shapes of positions of t?

3. (6 points) Give an example of a binary relation on $T(\mathcal{F}, \mathcal{V})$, which is compatible with $\mathcal{F}$-operations but is not closed under substitutions.

4. (6 points) Give an example of a binary relation on $T(\mathcal{F}, \mathcal{V})$, which is closed under $\mathcal{F}$-operations but is not compatible with $\mathcal{F}$-operations.

5. (4 points) Let $\mathcal{F} = \{0, s, add, mult\}$, where 0 is a constant, s is unary, and add and $mult$ are binary. Let σ be a $T(\mathcal{F}, \mathcal{V})$-substitution $\{x \mapsto s(y), y \mapsto add(s(0), s(s(0))), z \mapsto 0\}$. Compute $\sigma(t)$ for the following terms t:

(a) x.

(b) $add(x, add(y, add(z, s(0))))$.

(c) $mult(z, s(x))$.

(d) $add(s(x), mult(x, s(y)))$.

(e) $mult(s(u), add(u, x))$.

6. (8 points) Consider the set of identities

$$E = \{f(x) \approx x,\ f(f(a)) \approx g(x, x),\ g(x, f(x)) \approx b\}.$$

Which of the following statements is correct? Justify your answer.

(a) $E \vdash a \approx b$.

(b) $E \vdash g(x, y) \approx g(y, x)$.

(c) $E \vdash g(f(a), a) \approx f(b)$.

7. (8 points) Consider the set of identities which give an equational specification of groups:

$$E = \{x * (y * z) \approx (x * y) * z,\ x * e \approx x,\ x * i(x) \approx e\}$$

Show that

(a) $x \overset{*}{\leftrightarrow}_E e * x$.

(b) $f(i(a), i(b)) \overset{*}{\leftrightarrow}_E i(f(b, a))$.

(c) $E \vdash e \approx i(x) * x$.

(d) $E \vdash i(i(i(x)) * (x * i(x * e))) \approx i(i(i(i(x)))))$

8. (8 points, Ex. 3.7 from the book, [BN3.7]) Consider the set of identities:

$$E = \{f(x, f(y, z)) \approx f(f(x, y), z),\ f(f(x, y), x) \approx x\}$$

Show that $f(f(x, y), z) \overset{*}{\leftrightarrow}_E f(x, z)$. Show also the corresponding derivation $E \vdash f(f(x, y), x) \approx x$.

9. (9 points) Consider the set of $\{0, s, add\}$-identities which give an equational specification of addition over natural numbers:

$$E = \{add(0, x) \approx x,\ add(s(x), y) \approx s(add(x, y))\}.$$

Show that $E \not\models add(x, y) \approx add(y, x)$ (i.e., construct a model of E in which add is not commutative).

Part 3. Equational Problems

1. (4 points) Consider the following convergent TRS:

$$R = \{f(f(x)) \to x,\ g(x) \to x\}$$

Show that $f(f(f(g(f(f(f(g(f(g(x)))))))))) \approx_R f(g(f(g(f(x)))))$.

2. (7 points) Let $\mathcal{F} = \{i, j, k, l, m, f, g\}$, where i, j, k, l, m are constants and f and g are unary function symbols. Let G be the set of ground $\mathcal{F}$-identities:

$$G = \{i \approx j,\ k \approx l,\ f(i) \approx g(k),\ j \approx f(j),\ m \approx g(l)\}.$$

Using the congruence closure algorithm discussed in the lecture, prove that $f(m) \approx_G g(k)$.

3. (7 points) Let $\mathcal{F} = \{a, b, c, d, f\}$, where a, b, c, d are constants and f is a unary function symbol. Let G be the set of ground $\mathcal{F}$-identities:

$$G = \{a \approx c,\ d \approx f(f(c)),\ f(c) \approx f(f(f(b))),\ f(b) \approx a\}$$

(a) Show via $\overset{*}{\leftrightarrow}_G$ that $d \approx_G f(c)$ holds.

(b) Show via congruence closure that $d \approx_G f(c)$ holds.

4. (5 points) Let $\mathcal{F} = \{a, f\}$, where a is a constant and f is a binary function symbol. Show the steps of the unification algorithm for the following $T(\mathcal{F}, \mathcal{V})$-terms s and t:

(a) $s = f(y, z)$, $t = f(a, x)$

(b) $s = f(x, a)$, $t = f(a, x)$

(c) $s = f(y, y)$, $t = f(a, x)$

(d) $s = f(x, y)$, $t = f(f(x, y), a)$

5. (5 points) Show the steps of the matching algorithm for the terms s and t above, to match s to t.