Introduction to Unification Theory
Solving Systems of Linear Diophantine Equations

Temur Kutsia

RISC, Johannes Kepler University of Linz, Austria
kutsia@risc.jku.at
ACU-Unification

- We saw an example how to solve ACU-unification problem.
- Reduction to systems of linear Diophantine equations (LDEs) over natural numbers.
Elementary ACU-Unification

- Elementary ACU-unification problem

\[ \{ f(x, f(x, y)) \overset{?}{=}_{ACU} f(z, f(z, z)) \} \]

reduces to homogeneous linear Diophantine equation

\[ 2x + y = 3z. \]

- Each equation in the unification problem gives rise to one linear Diophantine equation.

- A most general ACU-unifier is obtained by combining all the unifiers corresponding to the minimal solutions of the system of LDEs.
Elementary ACU-Unification

- $\Gamma = \{ f(x, f(x, y)) = ?_{ACU} f(z, f(z, z)) \}$ and $S = \{ 2x + y = 3z \}$.
- $S$ has three minimal solutions: $(1, 1, 1), (0, 3, 1), (3, 0, 2)$.
- Three unifiers of $\Gamma$:
  
  $\sigma_1 = \{ x \mapsto v_1, y \mapsto v_1, z \mapsto v_1 \}$
  
  $\sigma_2 = \{ x \mapsto e, y \mapsto f(v_2, f(v_2, v_2)), z \mapsto v_2 \}$
  
  $\sigma_3 = \{ x \mapsto f(v_3, f(v_3, v_3)), y \mapsto e, z \mapsto f(v_3, v_3) \}$

- A most general unifier of $\Gamma$:
  
  $\sigma = \{ x \mapsto f(v_1, f(v_3, f(v_3, v_3))), y \mapsto f(v_1, f(v_2, f(v_2, v_2))), 
  
  z \mapsto f(v_1, f(v_2, f(v_3, v_3))) \}$
ACU-Unification with constants

- ACU-unification problem with constants

\[
\Gamma = \{ f(x, f(x, y)) \overset{?}{=} \text{ACU} f(a, f(z, f(z, z))) \}
\]

reduces to inhomogeneous linear Diophantine equation

\[
S = \{ 2x + y = 3z + 1 \}.
\]

- The minimal nontrivial natural solutions of \( S \) are \((0, 1, 0)\) and \((2, 0, 1)\).
ACU-Unification with constants

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\[ \Gamma = \{ f(x, f(x, y)) \overset{?}{=} \text{ACU} f(a, f(z, f(z, z))) \} \]

reduces to inhomogeneous linear Diophantine equation

\[ S = \{ 2x + y = 3z + 1 \}. \]

- Every natural solution of \( S \) is obtained by as the sum of one of the minimal solution and a solution of the corresponding homogeneous LDE \( 2x + y = 3z \).

- One element of the minimal complete set of unifiers of \( \Gamma \) is obtained from the combination of one minimal solution of \( S \) with the set of all minimal solutions of \( 2x + y = 3z \).
ACU-Unification with constants

- ACU-unification problem with constants

\[ \Gamma = \{ f(x, f(x, y)) \equiv_{ACU} f(a, f(z, f(z, z))) \} \]

reduces to inhomogeneous linear Diophantine equation

\[ S = \{ 2x + y = 3z + 1 \}. \]

- The minimal complete set of unifiers of \( \Gamma \) is \( \{ \sigma_1, \sigma_2 \} \), where

\[ \sigma_1 = \{ x \mapsto f(v_1, f(v_3, f(v_3, v_3))), y \mapsto f(a, f(v_1, f(v_2, f(v_2, v_2)))), z \mapsto f(v_1, f(v_2, f(v_3, v_3))) \} \]

\[ \sigma_2 = \{ x \mapsto f(a, f(a, f(v_1, f(v_3, f(v_3, v_3)))), y \mapsto f(v_1, f(v_2, f(v_2, v_2))), z \mapsto f(a, f(v_1, f(v_2, f(v_3, v_3)))) \} \]
How to Solve Systems of LDEs over Naturals?

Contejean-Devie Algorithm:

How to Solve Systems of LDEs over Naturals?

Contejean-Devie Algorithm:

Evelyne Contejean and Hervé Devie.
An Efficient Incremental Algorithm for Solving Systems of Linear Diophantine Equations.

Generalizes Fortenbacher’s Algorithm for solving a single equation:

Michael Clausen and Albrecht Fortenbacher.
Efficient Solution of Linear Diophantine Equations.
Homogeneous linear Diophantine system with $m$ equations and $n$ variables:

\[
\begin{aligned}
& a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\
& \vdots \\
& a_{m1}x_1 + \cdots + a_{mn}x_n = 0
\end{aligned}
\]

- $a_{ij}$'s are integers.
- Looking for nontrivial natural solutions.
Homogeneous Case

Example

\[
\begin{align*}
-x_1 + x_2 + 2x_3 - 3x_4 &= 0 \\
-x_1 + 3x_2 - 2x_3 - x_4 &= 0
\end{align*}
\]

Nontrivial solutions:

- \( s_1 = (0, 1, 1, 1) \)
- \( s_2 = (4, 2, 1, 0) \)
- \( s_3 = (0, 2, 2, 2) \)
- \( s_4 = (8, 4, 2, 0) \)
- \( s_5 = (4, 3, 2, 1) \)
- \( s_6 = (8, 5, 3, 1) \)
- \( \ldots \)
Homogeneous Case

Example

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Nontrivial solutions:

\begin{itemize}
  \item $s_1 = (0, 1, 1, 1)$
  \item $s_2 = (4, 2, 1, 0)$
  \item $s_3 = (0, 2, 2, 2) = 2s_1$
  \item $s_4 = (8, 4, 2, 0) = 2s_2$
  \item $s_5 = (4, 3, 2, 1) = s_1 + s_2$
  \item $s_6 = (8, 5, 3, 1) = s_1 + 2s_2$
  \item $\ldots$
\end{itemize}
Homogeneous Case

Homogeneous linear Diophantine system with $m$ equations and $n$ variables:

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& \quad \vdots \\
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- $a_{ij}$'s are integers.
- Looking for a basis in the set of nontrivial natural solutions.
Homogeneous linear Diophantine system with $m$ equations and $n$ variables:

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\begin{cases}
    a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\
    \vdots \\
    a_{m1}x_1 + \cdots + a_{mn}x_n = 0
\end{cases}
\]

- $a_{ij}$’s are integers.
- Looking for a basis in the set of nontrivial natural solutions.
- Does it exist?
Homogeneous Case

The basis in the set $S$ of nontrivial natural solutions of a homogeneous LDS is the set of $\gg$-minimal elements $S$.

$\gg$ is the ordering on tuples of natural numbers:

$$(x_1, \ldots, x_n) \gg (y_1, \ldots, y_n)$$

if and only if

- $x_i \geq y_i$ for all $1 \leq i \leq n$ and
- $x_i > y_i$ for some $1 \leq i \leq n$. 

Matrix Form

Homogeneous linear Diophantine system with $m$ equations and $n$ variables:

$$Ax_\downarrow = 0_\downarrow,$$

where

$$A := \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \quad x_\downarrow := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad 0_\downarrow := \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$
Matrix Form

- Canonical basis in $\mathbb{N}^n$: $(e_1, \ldots, e_n)$.

- $e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$, with 1 in $j$’s row.

- Then $Ax = x_1 A e_1 + \cdots + x_n A e_n$. 

$a$: The linear mapping associated to $A$. 

Then $a(x) = x_1 a(e_1) + \cdots + x_n a(e_n)$. 
Matrix Form

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Then $a(x) = x_1 a(e_1) + \cdots + x_n a(e_n)$. 
Case \( m = 1 \): Single homogeneous LDE \( a_1 x_1 + \cdots + a_n x_n = 0 \).

Fortenbacher’s idea:

- Search minimal solutions starting from the elements in the canonical basis of \( \mathbb{N}^n \).
- Suppose the current vector \( v_\downarrow \) is not a solution.
- It can be nondeterministically increased, component by component, until it becomes a solution or greater than a solution.
- To decrease the search space, the following restrictions can be imposed:
  - If \( a(v_\downarrow) > 0 \), then increase by one some \( v_j \) with \( a_j < 0 \).
  - If \( a(v_\downarrow) < 0 \), then increase by one some \( v_j \) with \( a_j > 0 \).
Case $m = 1$: Single homogeneous LDE $a_1 x_1 + \cdots + a_n x_n = 0$

Fortenbacher's idea:

- Search minimal solutions starting from the elements in the canonical basis of $\mathbb{N}^n$.
- Suppose the current vector $v_\downarrow$ is not a solution.
- It can be nondeterministically increased, component by component, until it becomes a solution or greater than a solution.
- To decrease the search space, the following restrictions can be imposed:
  - If $a(v_\downarrow) > 0$, then increase by one some $v_j$ with $a_j < 0$.
  - If $a(v_\downarrow) < 0$, then increase by one some $v_j$ with $a_j > 0$.
  - (If $a(v_\downarrow)a(e_j\downarrow) < 0$ for some $j$, increase $v_j$ by one.)
Fortenbacher’s condition
If $a(v_j)a(e_j) < 0$ for some $j$, increase $v_j$ by one.

Increasing $v_j$ by one: $a(v_j + e_j) = a(v_j) + a(e_j)$.

Going to the “right direction”, towards the origin.
Case $m = 1$: Single homogeneous LDE $a_1 x_1 + \cdots + a_n x_n = 0$.
Fortenbacher’s algorithm:
Case $m = 1$: Single homogeneous LDE $a_1x_1 + \cdots + a_nx_n = 0$.

Fortenbacher’s algorithm:

- Start with the pair $P, M$ of the set of potential solutions $P = \{e_1 \downarrow, \ldots, e_n \downarrow\}$ and the set of minimal nontrivial solutions $M = \emptyset$. 
Case $m = 1$: Single homogeneous LDE $a_1 x_1 + \cdots + a_n x_n = 0$. Fortenbacher’s algorithm:

- Start with the pair $P, M$ of the set of potential solutions $P = \{e_1, \ldots, e_n\}$ and the set of minimal nontrivial solutions $M = \emptyset$.
- Apply repeatedly the rules:
Case $m = 1$: Single homogeneous LDE $a_1 x_1 + \cdots + a_n x_n = 0$.

Fortenbacher’s algorithm:

- Start with the pair $P, M$ of the set of potential solutions $P = \{e_1, \ldots, e_n\}$ and the set of minimal nontrivial solutions $M = \emptyset$.

- Apply repeatedly the rules:

  1. $\{v\} \cup P', M \Rightarrow P', M$, if $v \gg u$ for some $u \in M$.
Case $m = 1$: Single homogeneous LDE $a_1 x_1 + \cdots + a_n x_n = 0$.

Fortenbacher's algorithm:

- Start with the pair $P, M$ of the set of potential solutions $P = \{ e_1 \downarrow, \ldots, e_n \downarrow \}$ and the set of minimal nontrivial solutions $M = \emptyset$.

- Apply repeatedly the rules:
  
  1. $\{v \downarrow \} \cup P', M \Rightarrow P', M$, if $v \downarrow \gg u \downarrow$ for some $u \downarrow \in M$.
  
  2. $\{v \downarrow \} \cup P', M \Rightarrow P', \{v \downarrow \} \cup M$, if $a(v \downarrow) = 0$ and rule 1 is not applicable.
Case $m = 1$: Single homogeneous LDE $a_1x_1 + \cdots + a_nx_n = 0$.

Fortenbacher’s algorithm:

- Start with the pair $P, M$ of the set of potential solutions $P = \{e_1, \ldots, e_n\}$ and the set of minimal nontrivial solutions $M = \emptyset$.

- Apply repeatedly the rules:

  1. $\{v\} \cup P', M \rightarrow P', M$, if $v \gg u$ for some $u \in M$.

  2. $\{v\} \cup P', M \rightarrow P', \{v\} \cup M$, if $a(v) = 0$ and rule 1 is not applicable.

  3. $P, M \rightarrow \{v + e_j | v \in P, a(v)a(e_j) < 0, j \in 1..n\}, M$, if rules 1 and 2 are not applicable.
Single Equation: Algorithm

Case $m = 1$: Single homogeneous LDE $a_1 x_1 + \cdots + a_n x_n = 0$.
Fortenbacher’s algorithm:

- Start with the pair $P, M$ of the set of potential solutions $P = \{e_1 \downarrow, \ldots, e_n \downarrow\}$ and the set of minimal nontrivial solutions $M = \emptyset$.

- Apply repeatedly the rules:
  
  1. $\{v \downarrow\} \cup P', M \Rightarrow P', M$, if $v \downarrow \gg u \downarrow$ for some $u \downarrow \in M$.
  
  2. $\{v \downarrow\} \cup P', M \Rightarrow P', \{v \downarrow\} \cup M$, if $a(v \downarrow) = 0$ and rule 1 is not applicable.

  3. $P, M \Rightarrow \{v \downarrow + e_j \downarrow \mid v \downarrow \in P, a(v \downarrow)a(e_j \downarrow) < 0, j \in 1..n\}, M$, if rules 1 and 2 are not applicable.

- If $\emptyset, M$ is reached, return $M$. 
System of Equations: Idea

- General case: System of homogeneous LDEs.
- $a(x_{\downarrow}) = 0_{\downarrow}$.
- Generalizing Fortenbacher’s idea:
  - Search minimal solutions starting from the elements in the canonical basis of $\mathbb{N}^n$.
  - Suppose the current vector $v_{\downarrow}$ is not a solution.
  - It can be nondeterministically increased, component by component, until it becomes a solution or greater than a solution.
  - To decrease the search space, increase only those components that lead to the “right direction”.
System of Equations: How to Restrict

- “Right direction”: Towards the origin.
- If $a(v_\downarrow) \neq 0_\downarrow$, then do $a(v_\downarrow + e_{j_\downarrow}) = a(v_\downarrow) + a(e_{j_\downarrow})$.
- $a(v_\downarrow) + a(e_{j_\downarrow})$ should lie in the half-space containing $O$.
- **Contejean-Devie condition**: If $a(v_\downarrow) \cdot a(e_{j_\downarrow}) < 0$ for some $j$, increase $v_j$ by one. ($\cdot$ is the scalar product.)
How to Restrict: Comparison

▶ Fortenbacher’s condition
If $a(v_{\downarrow}) a(e_{j\downarrow}) < 0$ for some $j$, increase $v_j$ by one.

▶ Contejean-Devie condition
If $a(v_{\downarrow}) \cdot a(e_{j\downarrow}) < 0$ for some $j$, increase $v_j$ by one.
System of Equations: Algorithm

System of homogeneous LDEs: \( a(x_\downarrow) = 0 \downarrow \).
Contejean-Devie algorithm:

- Start with the pair \( P, M \) where
  - \( P = \{ e_{1\downarrow}, \ldots, e_{n\downarrow} \} \) is the set of potential solutions,
  - \( M = \emptyset \) is the set of minimal nontrivial solutions.

- Apply repeatedly the rules:
  1. \( \{ v_\downarrow \} \cup P', M \rightarrow P', M \),
     if \( v_\downarrow \gg u_\downarrow \) for some \( u_\downarrow \in M \).
  2. \( \{ v_\downarrow \} \cup P', M \rightarrow P', \{ v_\downarrow \} \cup M, \)
     if \( a(v_\downarrow) = 0_\downarrow \) and rule 1 is not applicable.
  3. \( P, M \rightarrow \{ v_\downarrow + e_{j\downarrow} \mid v_\downarrow \in P, a(v_\downarrow) \cdot a(e_{j\downarrow}) < 0, j \in 1..n \}, M, \)
     if rules 1 and 2 are not applicable.

- If \( \emptyset, M \) is reached, return \( M \).
Contejean-Devie Algorithm on an Example

\[
\begin{align*}
- x_1 + x_2 + 2x_3 - 3x_4 &= 0 \\
- x_1 + 3x_2 - 2x_3 - x_4 &= 0
\end{align*}
\]

\[e_1 \downarrow = (1, 0, 0, 0)^T \quad e_2 \downarrow = (0, 1, 0, 0)^T\]
\[e_3 \downarrow = (0, 0, 1, 0)^T \quad e_4 \downarrow = (0, 0, 0, 1)^T\]

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   \(a(v\downarrow) \cdot a(e_j \downarrow) < 0, \ j \in 1..n\}, M,\)
   if rules 1 and 2 are not applicable.
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- x_1 &+ x_2 + 2x_3 - 3x_4 = 0 \\
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3. \(P, M \rightarrow \{v\downarrow + e_j\downarrow \mid v\downarrow \in P, \)
   \(a(v\downarrow) \cdot a(e_j\downarrow) < 0, j \in 1..n\}, M\),
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   \(a(v \downarrow) \cdot a(e_j \downarrow) < 0, j \in 1..n\}, M\),
   if rules 1 and 2 are not applicable.
Contejean-Devie Algorithm on an Example

\[
\begin{align*}
- x_1 + x_2 + 2x_3 - 3x_4 &= 0 \\
- x_1 + 3x_2 - 2x_3 - x_4 &= 0
\end{align*}
\]

\[e_1\downarrow = (1, 0, 0, 0)^T \quad e_2\downarrow = (0, 1, 0, 0)^T \]
\[e_3\downarrow = (0, 0, 1, 0)^T \quad e_4\downarrow = (0, 0, 0, 1)^T \]

1. \(\{v\downarrow\} \cup P', M \Rightarrow P', M\),
   if \(v\downarrow \gg u\downarrow\) for some \(u\downarrow \in M\).

2. \(\{v\downarrow\} \cup P', M \Rightarrow P', \{v\downarrow\} \cup M\),
   if \(a(v\downarrow) = 0\downarrow\) and rule 1 is not applicable.

3. \(P, M \Rightarrow \{v\downarrow + e_j\downarrow \mid v\downarrow \in P,\ a(v\downarrow) \cdot a(e_j\downarrow) < 0, j \in 1..n\}, M\),
   if rules 1 and 2 are not applicable.
Contejean-Devie Algorithm on an Example

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1. \( \{v \downarrow\} \cup P', M \implies P', M, \)
   if \( v \downarrow \gg u \downarrow \) for some \( u \downarrow \in M \).

2. \( \{v \downarrow\} \cup P', M \implies P', \{v \downarrow\} \cup M, \)
   if \( a(v \downarrow) = 0 \downarrow \) and rule 1 is not applicable.

3. \( P, M \implies \{v \downarrow + e_j \downarrow \mid v \downarrow \in P, \)
   \( a(v \downarrow) \cdot a(e_j \downarrow) < 0, j \in 1..n\}, M, \)
   if rules 1 and 2 are not applicable.
Contejean-Devie Algorithm on an Example

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\begin{aligned}
- x_1 + x_2 + 2x_3 - 3x_4 &= 0 \\
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\[e_1 \downarrow = (1, 0, 0, 0)^T \quad e_2 \downarrow = (0, 1, 0, 0)^T \]
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1. \(\{v \downarrow\} \cup P', M \rightarrow P', M\), if \(v \downarrow \gg u \downarrow\) for some \(u \downarrow \in M\).

2. \(\{v \downarrow\} \cup P', M \rightarrow P', \{v \downarrow\} \cup M\), if \(a(v \downarrow) = 0\) and rule 1 is not applicable.

3. \(P, M \rightarrow \{v \downarrow + e_j \downarrow \mid v \downarrow \in P\), \(a(v \downarrow) \cdot a(e_j \downarrow) < 0, j \in 1..n\}\), \(M\), if rules 1 and 2 are not applicable.
Contejean-Devie Algorithm on an Example

\[
\begin{align*}
- x_1 + x_2 + 2x_3 - 3x_4 &= 0 \\
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\[
e_1 = (1, 0, 0, 0)^T \\
e_2 = (0, 1, 0, 0)^T \\
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e_4 = (0, 0, 0, 1)^T
\]

1. \( \{v\} \cup P', M \implies P', M, \) if \( v \gg u \) for some \( u \in M \).

2. \( \{v\} \cup P', M \implies P', \{v\} \cup M, \) if \( a(v) = 0 \) and rule 1 is not applicable.

3. \( P, M \implies \{v + e_j | v \in P, \ a(v) \cdot a(e_j) < 0, j \in 1..n\}, M, \) if rules 1 and 2 are not applicable.
Contejean-Devie Algorithm on an Example

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\begin{align*}
- x_1 + x_2 + 2x_3 - 3x_4 &= 0 \\
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\[e_1 \downarrow = (1, 0, 0, 0)^T \quad e_2 \downarrow = (0, 1, 0, 0)^T \]
\[e_3 \downarrow = (0, 0, 1, 0)^T \quad e_4 \downarrow = (0, 0, 0, 1)^T\]

1. \(\{v \downarrow\} \cup P', M \Rightarrow P' \cup M\), if \(v \downarrow \gg u \downarrow\) for some \(u \downarrow \in M\).

2. \(\{v \downarrow\} \cup P', M \Rightarrow P' \cup \{v \downarrow\} \cup M\), if \(a(v \downarrow) = 0 \downarrow\) and rule 1 is not applicable.

3. \(P, M \Rightarrow \{v \downarrow + e_j \downarrow \mid v \downarrow \in P, a(v \downarrow) \cdot a(e_j \downarrow) < 0, j \in 1..n\}\), if rules 1 and 2 are not applicable.
Contejean-Devie Algorithm on an Example

\[
\begin{align*}
- x_1 + x_2 + 2x_3 - 3x_4 &= 0 \\
- x_1 + 3x_2 - 2x_3 - x_4 &= 0
\end{align*}
\]
Properties of the Algorithm

- Completeness
- Soundness
- Termination

In the theorems:

\[ a(x_\downarrow) = 0_\downarrow: \text{An } n\text{-variate system of homogeneous LDEs.} \]

\((e_1\downarrow, \ldots, e_n\downarrow): \text{The canonical basis of } \mathbb{N}^n.\)

\(\mathcal{B}(a(x_\downarrow) = 0_\downarrow): \text{Basis in the set of nontrivial natural solutions of} \]

\[ a(x_\downarrow) = 0_\downarrow. \]

\(\|v_\downarrow\|: \text{Euclidean norm of } v_\downarrow.\)
Properties of the Algorithm

Theorem (Completeness)

Let \((e_1\downarrow, \ldots, e_n\downarrow), \emptyset \longmapsto^* \emptyset, M\) be the sequence of transformations performed by the Contejean-Devie algorithm for \(a(x\downarrow) = 0\downarrow\). Then

\[ B(a(x\downarrow) = 0\downarrow) \subseteq M. \]
Properties of the Algorithm

Theorem (Soundness)

Let \((e_1 \downarrow, \ldots, e_n \downarrow), \emptyset \Longrightarrow^* \emptyset, M\) be the sequence of transformations performed by the Contejean-Devie algorithm for \(a(x\downarrow) = 0\downarrow\). Then

\[ M \subseteq \mathcal{B}(a(x\downarrow) = 0\downarrow). \]
Properties of the Algorithm

Lemma (Limit Lemma)

Let \( v_1 \downarrow, v_2 \downarrow, \ldots \) be an infinite sequence satisfying the Contejean-Devie condition for \( a(x \downarrow) = 0 \downarrow \):

- \( v_1 \downarrow \) is a basic vector and for each \( i \geq 1 \) there exists \( 1 \leq j \leq n \) such that \( a(v_i \downarrow) \cdot a(e_j \downarrow) < 0 \) and \( v_{i+1} \downarrow = v_i \downarrow + e_j \downarrow \).

Then

\[
\lim_{k \to \infty} \frac{\| a(v_k \downarrow) \|}{k} = 0
\]

Theorem (Termination)

Let \( v_1 \downarrow, v_2 \downarrow, \ldots \) be an infinite sequence satisfying the conditions of the Limit Lemma. Then there exist \( v \downarrow \) and \( k \) such that

- \( v \downarrow \) is a solution of \( a(x \downarrow) = 0 \downarrow \), and

- \( v \downarrow \ll v_k \downarrow \).
Non-Homogeneous Case

Non-homogeneous linear Diophantine system with \( m \) equations and \( n \) variables:

\[
\begin{align*}
\begin{cases}
    a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
    \vdots & \vdots & \vdots \\
    a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m
\end{cases}
\end{align*}
\]

- \( a \)'s and \( b \)'s are integers.
- Matrix form: \( a(x_\downarrow) = b_\downarrow \).
Non-Homogeneous Case. Solving Idea

Turn the system into a homogeneous one, denoted $S_0$:

$$
\begin{align*}
-b_1 x_0 &+ a_{11} x_1 + \cdots + a_{1n} x_n = 0 \\
\vdots \\
-b_m x_0 &+ a_{m1} x_1 + \cdots + a_{mn} x_n = 0
\end{align*}
$$

- Solve $S_0$ and keep only the solutions with $x_0 \leq 1$.
- $x_0 = 1$: a minimal solution for $a(x_\downarrow) = b_\downarrow$.
- $x_0 = 0$: a minimal solution for $a(x_\downarrow) = 0_\downarrow$.
- Any solution of the non-homogeneous system $a(x_\downarrow) = b_\downarrow$ has the form $x_\downarrow + y_\downarrow$ where:
  - $x_\downarrow$ is a minimal solution of $a(x_\downarrow) = b_\downarrow$.
  - $y_\downarrow$ is a linear combination (with natural coefficients) of minimal solutions of $a(x_\downarrow) = 0_\downarrow$. 