Introduction to Unification Theory

Narrowing

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Overview

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- The most important special case of the $E$-unification problem, when the equational theory can be represented by a ground convergent set of rewrite rules.
- Narrowing: The process that is used to solve such $E$-unification problems.
Let $E$ be a set of identities, and $R$ a convergent term rewriting equivalent to $E$.

$\sigma$ is an $E$-unifier of $s$ and $t$, then $s\sigma$ and $t\sigma$ have the same $R$-normal forms.

Idea: Construct the unifier and the corresponding reduction chains simultaneously.
Example

- \( E = \{0 + x = x\} \), \( R = \{0 + x \rightarrow x\} \).
- Solve \( E \)-unification problem \( \{y + z \doteq_{E} 0\} \).
- Proceed as follows:
  1. Look for an instance of \( y + z \) to which the rewrite rule applies. Such instance is computed by syntactically unifying \( y + z \) and \( 0 + x \), yielding the mgu \( \varphi = \{y \mapsto 0, z \mapsto x\} \).
  2. \((y + z)\varphi = 0 + x\), rewriting it with \( 0 + x \rightarrow x \) gives \( x \) and we obtain a new problem \( \{x \doteq_{E} 0\} \).
  3. \( \{x \doteq_{E} 0\} \) has the syntactic mgu \( \vartheta = \{x \mapsto 0\} \).
  4. By this process we have simultaneously constructed the \( E \)-unifier \( \sigma = \varphi\vartheta = \{y \mapsto 0, z \mapsto 0, x \mapsto 0\} \) and the rewrite chain \((y + z)\sigma = 0 + 0 \rightarrow 0 = 0\sigma\).
Preliminaries

- **A rewrite rule**: a directed equation $l \rightarrow r$, where $\text{vars}(r) \subseteq \text{vars}(l)$.
- **A term rewriting system (TRS)**: a set of rewrite rules.
- $s|_p$: The subterm of $s$ at position $p$.
- $s[t]|_p$: A term obtained from $s$ by replacing its subterm at position $p$ with the term $t$.
- The **rewrite relation** $R$ associated with a TRS $R$: $s \rightarrow_R t$ if there exists a variant $l \rightarrow r$ of a rewrite rule in $R$, a position $p$ in $s$, and a substitution $\sigma$ such that $s|_p = l\sigma$ and $t = s[r\sigma]|_p$.
- $s|_p$ is called a **redex**.
Preliminaries

- \( \rightarrow_R \): The transitive-reflexive closure of \( \rightarrow_R \).
- \( s \) reduces to \( t \) in \( R \): \( s \rightarrow_R t \).
- If \( E \) is the set of equations corresponding to \( R \), i.e.,
  \[ E = \{ l \equiv r \mid l \rightarrow r \in R \} \],
  then \( \models_E \) coincides with the reflexive-symmetric-transitive closure of \( R \).
- Two terms \( t_1, t_2 \) are joinable (wrt \( R \)), denoted by \( t_1 \downarrow_R t_2 \), if
  there exists a term \( s \) such that \( t_1 \rightarrow_R s \) and \( t_2 \rightarrow_R s \).
- A term \( s \) is a normal form (wrt \( R \)) if there is no term \( t \) with
  \( s \rightarrow_R t \).
Preliminaries

- $R$ is **terminating** if there are no infinite reduction sequences $t_1 \rightarrow_R t_2 \rightarrow_R t_3 \rightarrow_R \cdots$.
- $R$ is **confluent** if for all terms $s, t_1, t_2$ with $s \rightarrow_R t_1$ and $s \rightarrow_R t_2$ we have $t_1 \Downarrow_R t_2$.
- $R$ is **convergent** if it is confluent and terminating.
### Preliminaries

- A constraint system: either $\bot$ (representing failure) or a triple $P; C; S$.
- $P$: A multiset of equations, representing the schema of the problem.
- $C$: A set of equations, representing constraints on variables in $P$.
- $S$: A set of equations, representing bindings in the solution.
- $C$ plays the role similar to $P$ earlier, the rules from $\mathcal{U}$ will be applied to $C; S$ as before.
- $\vartheta$ is said to be a solution (or $E$-unifier) of a system $P; C; S$ if it $E$-unifies each equation in $P$, and unifies each of the equations in $C$ and $S$; the system $\bot$ has no $E$-unifiers.
Assumptions

- The rewrite system $R$ is ground convergent with respect to a reduction ordering $\succ$.

- $R$ is represented as a numbered sequence of rules

\[
\{l_1 \rightarrow r_1, \ldots, l_n \rightarrow r_n\}.
\]

- The index of a rule is its number in this sequence.
Preliminaries

Restricted form of substitution:

Definition
Given a rewrite system $R$, a substitution $\vartheta$ is $R$-reduced (or just reduced if $R$ is unimportant) if for every $x \in \text{dom}(\vartheta)$, $x$ is in $R$-normal form.

Example

$$R = \{f(f(x, y), z) \rightarrow f(x, f(y, z)), f(x, x) \rightarrow x\}.$$  
$$\vartheta_1 = \{x \mapsto f(f(u, v), w), y \mapsto f(a, f(a, a))\} : \text{not } R\text{-reduced}.$$  
$$\vartheta_2 = \{x \mapsto f(u, f(v, w)), y \mapsto a\} : R\text{-reduced}.$$  

For any $\vartheta$ and terminating set of rules $R$ one can find an $R$-equivalent reduced substitution $\vartheta'$. 
Outline

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The Calculus $\mathcal{B}$ for Basic Narrowing

The rule set $S$:

**Trivial:** $P; \{s \doteq ? s\} \cup C'; S \Longrightarrow P; C'; S.$

**Decomposition:** $P; \{f(s_1, \ldots, s_n) \doteq ? f(t_1, \ldots, t_n)\} \cup C'; S \Longrightarrow P; \{s_1 \doteq? t_1, \ldots, s_n \doteq? t_n\} \cup C'; S,$

where $n \geq 0$.

**Orient:** $P; \{t \doteq ? x\} \cup C'; S \Longrightarrow P; \{x \doteq ? t\} \cup C'; S$

if $t$ is not a variable.

**Basic Variable** $P; \{x \doteq ? t\} \cup C'; S \Longrightarrow$

**Elimination:** $P; C'\{x \mapsto t\}; S\{x \mapsto t\} \cup \{x \approx t\},$

if $x \notin \text{vars}(t)$. 
The Calculus $\mathcal{B}$ for Basic Narrowing

Two extra rules:

**Constrain:** $\{e\} \cup P'; C; S \xrightarrow{\text{Con}} P'; \{e\sigma_S\} \cup C'; S$.

**Lazy Paramodulation:**

$\{e[t]\} \cup P'; C; S \xrightarrow{\text{LP}}$

$\{e[r]\} \cup P'; \{l\sigma_S \overset{?}{=} t\sigma_S\} \cup C; S$,

for a fresh variant of $l \rightarrow r$ from $R$, where

- $e[t]$ is an equation where the term $t$ occurs,
- $t$ is not a variable,
- the top symbol of $l$ and $t$ are the same.
Theorem

Let $R$ be a ground convergent set of rewrite rules. If $P; \emptyset; \emptyset \rightarrow^*_B \emptyset; \emptyset; S$, then $\sigma_S$ is an $R$-unifier of $P$.

Proof.

Exercise.
Completeness of the Calculus $\mathcal{B}$

Theorem

Let $R$ be a ground convergent set of rewrite rules. If $\vartheta$ is an $R$-reduced solution of $P; \emptyset; \emptyset$, then there exists a sequence $P; \emptyset; \emptyset \xrightarrow{\star} \emptyset; \emptyset; S$ such that $\sigma_S \leq_R \vars(P) \vartheta$.

Proof.

- We may assume that $P\vartheta$ is ground and that $\vartheta$ is $R$-reduced, since the relation $\succ$ does not distinguish between $R$-equivalent substitutions.

- Thus, we will prove a stronger result, that when $\vartheta$ is $R$-reduced, then $\sigma_S \leq_{\vars(P)} \vartheta$. 
Completeness of the Calculus $\mathcal{B}$

Theorem

Let $R$ be a ground convergent set of rewrite rules. If $\vartheta$ is an $R$-reduced solution of $P; \emptyset; \emptyset$, then there exists a sequence $P; \emptyset; \emptyset \xrightarrow{\mathcal{B}} \emptyset; \emptyset; S$ such that $\sigma_S \leq^R_{\text{vars}(P)} \vartheta$.

Proof (cont.)

The complexity $\langle M, n_1, n_2, n_3 \rangle$ for $P; C; S$ and its solution $\vartheta$:

- $M =$ The multiset of all terms occurring in $P \vartheta$;
- $n_1 =$ The number of distinct variables in $C$;
- $n_2 =$ The number of symbols in $C$;
- $n_3 =$ The number of equations $t \vdash_{E} ? x \in C$ where $t$ is not a variable.

Associate to it the well-founded ordering: The multiset extension of $\prec$ for the first component, and the ordering on natural numbers on the remaining components.
Completeness of the Calculus $\mathcal{B}$

**Theorem**

Let $R$ be a ground convergent set of rewrite rules. If $\vartheta$ is an $R$-reduced solution of $P; \emptyset; \emptyset$, then there exists a sequence $P; \emptyset; \emptyset \xrightarrow{*_{\mathcal{B}}} \emptyset; \emptyset; S$ such that $\sigma_S \leq^R \varsigma(P) \vartheta$.

**Proof (cont.)**

Show by induction on this measure that if $\vartheta$ is a solution of $P; C; S'$ with $S'$ in a solved form, then there exists a sequence

$$P; C; S' \xrightarrow{*} \emptyset; \emptyset; S$$

such that $\sigma_S \leq^{\mathcal{X}} \vartheta$, where $\mathcal{X} = \varsigma(P, C, S')$.

The base case $\emptyset; \emptyset; S$ is trivial.
Completeness of the Calculus $B$

Theorem

Let $R$ be a ground convergent set of rewrite rules. If $\vartheta$ is an $R$-reduced solution of $P; \emptyset; \emptyset$, then there exists a sequence $P; \emptyset; \emptyset \xrightarrow{\ast}_B \emptyset; \emptyset; S$ such that $\sigma_S \leq_R \vartheta$.

Proof (cont.)

For the induction step there are several overlapping cases:

1. If $C = \{s :? t\} \cup C'$, then $sv\vartheta = tv\vartheta$ and we use $S$ to generate a transformation step to a smaller system containing the same set of variables, and with the same solution. By the induction hypothesis, we have

$$P; C; S' \xrightarrow{\mathcal{S}} P; C''; S'' \xrightarrow{\ast} \emptyset; \emptyset; S$$

such that $\sigma_S \leq_{\mathcal{X}} \vartheta$ for $\mathcal{X} = \text{vars}(P, C, S')$. 
Completeness of the Calculus $\mathcal{B}$

Theorem

Let $R$ be a ground convergent set of rewrite rules. If $\emptyset$ is an $R$-reduced solution of $P; \emptyset; \emptyset$, then there exists a sequence $P; \emptyset; \emptyset \longrightarrow^* \emptyset; \emptyset; S$ such that $\sigma_S \leq_R \vars(P) \emptyset$.

Proof (cont.)

2. If $P = \{s \leftarrow t\} \cup P'$ and $s\emptyset = t\emptyset$, then we may apply Constrain to obtain a smaller system (reducing the component $M$) with the same solution and the same set of variables, and we conclude as in the previous case.
Completeness of the Calculus $\mathcal{B}$

Theorem

Let $R$ be a ground convergent set of rewrite rules. If $\vartheta$ is an $R$-reduced solution of $P; \emptyset; \emptyset$, then there exists a sequence $P; \emptyset; \emptyset \Longrightarrow^* \emptyset; \emptyset; S$ such that $\sigma_S \leq^R_{\text{vars}(P)} \vartheta$.

Proof (cont.)

3. Assume $P = \{s \overset{?}{=} t\} \cup P'$ and there is an innermost redex in, say $s\vartheta$.
   - If more than one instance of a rule from $R$ reduces this redex, we choose the rule with the smallest index in the set $R$.
   - Since $\vartheta$ is $R$-reduced, the redex must occur inside the non-variable positions of $s$. 
Completeness of the Calculus $\mathcal{B}$

**Theorem**

Let $R$ be a ground convergent set of rewrite rules. If $\vartheta$ is an $R$-reduced solution of $P; \emptyset; \emptyset$, then there exists a sequence $P; \emptyset; \emptyset \xrightarrow{\ast} \emptyset; \emptyset; S$ such that $\sigma_S \leq^{\text{vars}(P)}_R \vartheta$.

**Proof (cont.)**

3. ▶ Hence, we have the transformation:

$$\{s[s'] \overset{?}{=} t \} \cup P'; C; S' \xrightarrow{\text{LP}}$$

$$\{s[r] \overset{?}{=} t \} \cup P'; \{l\sigma'_S \overset{?}{=} s'\sigma'_S \} \cup C; S'$$

▶ The new system smaller with respect to its new solution $\vartheta' = \vartheta \rho$. $\vartheta'$ is still $R$-reduced.

▶ By the induction hypothesis,

$$\{s[r] \overset{?}{=} t \} \cup P'; \{l\sigma'_S \overset{?}{=} s'\sigma'_S \} \cup C; S' \xrightarrow{\ast} \emptyset; \emptyset; S$$ such that $\sigma_S \leq^\mathcal{X} \vartheta'$ with $\mathcal{X} = \text{vars}(l, r, P, C, S')$, and since $x\vartheta = x\vartheta'$ for every $x \in \text{vars}(P, C, S')$, the induction is complete.
Example

- $R = \{0 + x \rightarrow x, s(x) + y \rightarrow s(x + y)\}$
- Goal: $\{z + z \vdash s(s(0))\}$
- Successful derivation:

\[
\begin{align*}
\{z + z \vdash s(s(0))\}; \emptyset; \emptyset & \Rightarrow_{LP} \\
\{s(x + y) \vdash s(s(0))\}; \{z + z \vdash s(x) + y\}; \emptyset & \Rightarrow_{D} \\
\{s(x + y) \vdash s(s(0))\}; \{z \vdash s(x), z \vdash y\}; \emptyset & \Rightarrow_{BVE} \\
\{s(x + y) \vdash s(s(0))\}; \{s(x) \vdash y\}; \{z \approx s(x)\} & \Rightarrow_{O} \\
\{s(x + y) \vdash s(s(0))\}; \{y \vdash s(x)\}; \{z \approx s(x)\} & \Rightarrow_{BVE} \\
\{s(x + y) \vdash s(s(0))\}; \emptyset; \{z \approx s(x), y \approx s(x)\} & \Rightarrow_{LP} \\
\{s(x') \vdash s(s(0))\}; \{x + s(x) \vdash 0 + x'\};& \\
\{z \approx s(x), y \approx s(x)\} & \Rightarrow_{D}
\end{align*}
\]
Example

- \( R = \{ 0 + x \rightarrow x, s(x) + y \rightarrow s(x + y) \} \)
- Goal: \( \{ z + z \doteq ? s(s(0)) \} \)
- Successful derivation (cont.):

\[
\{ s(x') \doteq ? s(s(0)) \}; \{ x \doteq ? 0, s(x) \doteq ? x' \}; \{ z \approx s(x), y \approx s(x) \} \implies_{\text{BVE}} \\
\{ s(x') \doteq ? s(s(0)) \}; \{ s(0) \doteq ? x' \}; \{ z \approx s(0), y \approx s(0), x \approx 0 \} \implies_{\text{O}} \\
\{ s(x') \doteq ? s(s(0)) \}; \{ x' \doteq ? s(0) \}; \{ z \approx s(0), y \approx s(0), x \approx 0 \} \implies_{\text{BVE}} \\
\{ s(x') \doteq ? s(s(0)) \}; \emptyset; \{ z \approx s(0), y \approx s(0), x \approx 0, x' \approx s(0) \} \implies_{\text{C}} \\
\emptyset; \{ s(s(0)) \doteq ? s(s(0)) \}; \{ z \approx s(0), y \approx s(0), x \approx 0, x' \approx s(0) \} \implies_{\text{T}} \\
\emptyset; \emptyset; \{ z \approx s(0), y \approx s(0), x \approx 0, x' \approx s(0) \}.
\]
Counterexample for Nonterminating $R$

If $R$ is not terminating, $B$ may not find solutions.

Counterexample by A. Middeldorp and E. Hamoen, 1994:

- $R = \{ f(x) \rightarrow g(x, x), a \rightarrow b, g(a, b) \rightarrow c, g(b, b) \rightarrow f(a) \}$
- Goal: $\{ f(a) \vdash c \}$
- The goal is unifiable ($f(a) \vdash c$), but $B$ can not verify it:

$$
\begin{align*}
\{ f(a) \vdash c \}; \emptyset; \emptyset & \rightarrow_{\text{LP}} \\
\{ g(x, x) \vdash c \}; \{ f(x) \vdash f(a) \}; \emptyset & \rightarrow_{\text{D}} \\
\{ g(x, x) \vdash c \}; \{ x \vdash a \}; \emptyset & \rightarrow_{\text{BVE}} \\
\{ g(x, x) \vdash c \}; \emptyset; \{ x \approx a \} & \rightarrow_{\text{C}} \\
\emptyset; \{ g(a, a) \vdash c \}; \{ x \approx a \} & \rightarrow \bot
\end{align*}
$$
Counterexample for Nonterminating $R$

If $R$ is not terminating, $B$ may not find solutions.

Counterexample by A. Middeldorp and E. Hamoen, 1994:

- $R = \{f(x) \rightarrow g(x, x), a \rightarrow b, g(a, b) \rightarrow c, g(b, b) \rightarrow f(a)\}$
- Goal: $\{f(a) \doteqdot c\}$
- Second unsuccessful derivation:

\[
\begin{align*}
\{f(a) \doteqdot c\}; \emptyset; \emptyset \rightarrow_{\text{LP}} \\
\{g(x, x) \doteqdot c\}; \{f(x) \doteqdot f(a)\}; \emptyset \rightarrow_{\text{D}} \\
\{g(x, x) \doteqdot c\}; \{x \doteqdot a\}; \emptyset \rightarrow_{\text{BVE}} \\
\{g(x, x) \doteqdot c\}; \emptyset; \{x \approx a\} \rightarrow_{\text{LP}} \\
\{c \doteqdot c\}; \{g(a, a) \doteqdot g(a, b)\}; \{x \approx a\} \rightarrow_{\text{D}} \\
\{c \doteqdot c\}; \{a \doteqdot b, a \doteqdot a\}; \{x \approx a\} \rightarrow \bot
\end{align*}
\]
Counterexample for Nonterminating $R$

If $R$ is not terminating, $B$ may not find solutions.

Counterexample by A. Middeldorp and E. Hamoen, 1994:

$R = \{f(x) \rightarrow g(x, x), a \rightarrow b, g(a, b) \rightarrow c, g(b, b) \rightarrow f(a)\}$

Goal: $\{f(a) \not\Rightarrow c\}$

Third unsuccessful derivation:

\[
\begin{align*}
\{f(a) \not\Rightarrow c\}; \emptyset; \emptyset & \Rightarrow_{LP} \{g(x, x) \not\Rightarrow c\}; \{f(x) \not\Rightarrow f(a)\}; \emptyset \Rightarrow_{D} \{g(x, x) \not\Rightarrow c\}; \{x \not\Rightarrow a\}; \emptyset \Rightarrow_{BVE} \{g(x, x) \not\Rightarrow c\}; \emptyset; \{x \approx a\} \Rightarrow_{LP} \{f(a) \not\Rightarrow c\}; \{g(a, a) \not\Rightarrow g(b, b)\}; \{x \approx a\} \Rightarrow_{D} \{f(a) \not\Rightarrow c\}; \{a \not\Rightarrow b\}; \{x \approx a\} \Rightarrow \bot
\end{align*}
\]
Counterexample for Nonterminating $R$

If $R$ is not terminating, $B$ may not find solutions.

Counterexample by A. Middeldorp and E. Hamoen, 1994:

1. $R = \{f(x) \rightarrow g(x, x), a \rightarrow b, g(a, b) \rightarrow c, g(b, b) \rightarrow f(a)\}$
2. Goal: $\{f(a) \not\Rightarrow c\}$
3. Fourth unsuccessful derivation:

\[
\begin{align*}
\{f(a) \not\Rightarrow c\}; \emptyset; \emptyset & \Rightarrow_{LP} \{f(b) \not\Rightarrow c\}; \emptyset; \emptyset \\
\{f(b) \not\Rightarrow c\}; \{a \not\Rightarrow a\}; \emptyset & \Rightarrow_{T} \{f(b) \not\Rightarrow c\}; \emptyset; \emptyset \\
\{g(x, x) \not\Rightarrow c\}; \{f(x) \not\Rightarrow f(b)\}; \emptyset & \Rightarrow_{D} \\
\{g(x, x) \not\Rightarrow c\}; \{x \not\Rightarrow b\}; \emptyset & \Rightarrow_{BVE} \\
\emptyset; \{g(b, b) \not\Rightarrow c\}; \{x \Rightarrow b\} & \Rightarrow_{C} \\
\emptyset; \{g(b, b) \not\Rightarrow c\}; \{x \Rightarrow b\} & \Rightarrow_{\bot}
\end{align*}
\]
Counterexample for Nonterminating $R$

If $R$ is not terminating, $B$ may not find solutions.

Counterexample by A. Middeldorp and E. Hamoen, 1994:

- $R = \{f(x) \rightarrow g(x, x), a \rightarrow b, g(a, b) \rightarrow c, g(b, b) \rightarrow f(a)\}$
- Goal: $\{f(a) \doteq c\}$
- An infinite derivation:

$$\{f(a) \doteq c\}; \emptyset; \emptyset \Rightarrow_{\text{LP}}$$
$$\{f(b) \doteq c\}; \{a \doteq a\}; \emptyset \Rightarrow_{\text{T}} \{f(b) \doteq c\}; \emptyset; \emptyset \Rightarrow_{\text{LP}}$$
$$\{g(x, x) \doteq c\}; \{f(x) \doteq f(b)\}; \emptyset \Rightarrow_{\text{D}}$$
$$\{g(x, x) \doteq c\}; \{x \doteq b\}; \emptyset \Rightarrow_{\text{BVE}}$$
$$\{g(x, x) \doteq c\}; \emptyset; \{x \approx b\} \Rightarrow_{\text{LP}}$$
$$\{f(a) \doteq c\}; \{g(b, b) \doteq g(b, b)\}; \{x \approx b\} \Rightarrow_{\text{T}}$$
$$\{f(a) \doteq c\}; \emptyset; \{x \approx b\} \Rightarrow \ldots$$
Strategies and refinements

- Variety of strategies and refinements can be developed for the basic narrowing calculus without destroying completeness.
- For instance, composite rules, simplification, redex orderings and variable abstraction.
- For more details, see, e.g.,

  F. Baader and W. Snyder. Unification theory.