(5) Formal Solutions

Question: Given a differential field K and $a_0, \dots, a_r \in K$, does there always exist an extension E of K such that the DEQ

$$a_0 y + \dots + a_r D^r(y) = 0$$

has a fundamental system in E? If so, what is the smallest such E? Is it uniquely determined by the equation?

For convenience, we discuss these questions for systems of DEQs. Given $A \in K^{r \times r}$, a solution is a vector $y = (y_1, \dots, y_r) \in E^r$ st

$$D(y) = A \cdot y.$$

Choosing

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \\ -\frac{a_0}{a_r} & \cdots & & \cdots & -\frac{a_{r-1}}{a_r} \end{pmatrix}$$

we have that $y = (y_0, D(y_0), \dots, D^{r-1}(y_0))$ is a solution of $D(y) = Ay \iff y_0$ is a solution of

$$a_0 y + a_1 D(y) + \dots + a_r D^r(y) = 0.$$

$Y \in E^{r\times r}$ is called a fundamental matrix for $A \in K^{r\times r}$ if Y is invertible (i.e. $\det Y \neq 0$) and $D(Y) = A\cdot Y$.

Ex: If A is as above, then $\{y_1 .. y_r\}$ is a fundamental system of

$$a_0 y + a_1 D(y) + \dots + a_r D^r(y) = 0$$

$$\Leftrightarrow \qquad Y = \begin{pmatrix} y_1 & \cdots & y_r \\ D(y_1) & \ddots & \vdots \\ \vdots & & \\ D^{r-1}(y_1) & \cdots & D^{r-1}(y_r) \end{pmatrix} \in E^{r\times r}$$

is a fundamental matrix for A.

Goal: Given $A \in K^{r\times r}$, construct a differential ring R containing K such that there exists a fundamental matrix $Y \in R^{r\times r}$ for A.

Easy: Take the polynomial ring

$$R = K[y_{11}, y_{12}, \dots, y_{1r}, \\ y_{21}, \dots \\ \vdots \\ y_{r1}, \dots, y_{rr}]$$

in r^2 variables and turn it into a

differential ring by extending D from K to R via

$$\begin{pmatrix} D(y_{11}) & \cdots & \\ \vdots & \ddots & \\ & & D(y_{rr}) \end{pmatrix} := A \cdot \begin{pmatrix} y_{11} & \cdots & \vdots \\ \vdots & \ddots & \vdots \\ & \cdots & y_{rr} \end{pmatrix}.$$

Then obviously R is a differential ring extension of K, and $Y = ((y_{ij}))$ is a fundamental matrix for A in $E = \mathrm{Quot}(R)$ because $\det Y$ is a nonzero polynomial in the new variables y_{ij}.

<u>Problem:</u> E might contain "fake constants".

<u>Ex:</u> $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $D(y_{ij}) = y_{ij}$ for all i,j.

Then $D\left(\frac{y_{ij}}{y_{kl}}\right) = \frac{D(y_{ij})\, y_{kl} - y_{ij}\, D(y_{kl})}{y_{kl}^2} = 0$, so

$y_{ij} / y_{kl} \in \mathrm{Const}(E) \setminus K$ when $(i,j) \neq (k,l)$.

<u>Algebraic analog:</u> $x^2 - 2 = 0$ has two solutions, but the splitting field cannot be constructed as $\mathbb{Q}[y_1, y_2] / \langle y_1^2 - 2, y_2^2 - 2 \rangle$ because this ring does not "know" whether $y_1 = +\sqrt{2}$, $y_2 = -\sqrt{2}$ or the other way around (or, even worse, $y_1 = y_2 = \sqrt{2}$)

Better: $\mathbb{Q}[y_1, y_2] / \langle y_1^2 - 2, y_2^2 - 2, y_1 + y_2 \rangle$.

<u>Idea:</u> Identify fake constants with elements of $C = \mathrm{Const}(K)$.

<u>Def:</u> If (R, D) is a differential ring and $\mathfrak{a} \subseteq R$ an ideal with $\forall a \in \mathfrak{a}: D(a) \in \mathfrak{a}$, then $\mathfrak{a}$ is called a differential ideal of R.

<u>Observe:</u> If $\mathfrak{a}$ is a differential ideal of R then we can turn the ring $\overline{R} = R/\mathfrak{a}$ into a differential ring $(\overline{R}, \overline{D})$ by setting

$$\overline{D}(a + \mathfrak{a}) := D(a) + \mathfrak{a}$$

because if $u \equiv v \mod \mathfrak{a}$, i.e. $u - v \in \mathfrak{a}$, then $D(u) - D(v) = D(u-v) \in \mathfrak{a}$, so $D(u) \equiv D(v) \mod \mathfrak{a}$, so the definition is meaningful.

<u>Ex:</u> $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathfrak{a} = \langle y_{11} - y_{12}, y_{11} - y_{21}, y_{11} - y_{22} \rangle$.
Then $R/\mathfrak{a} \cong K[y]$ with $y \mathrel{\hat{=}} e^x$. But now $Y = \begin{pmatrix} y & y \\ y & y \end{pmatrix}$ is no longer fundamental. Better: $\mathfrak{a} = \langle 2y_{11} - y_{12}, 3y_{11} - y_{21}, 4y_{11} - y_{22} \rangle$. Then $Y = \begin{pmatrix} y & 2y \\ 3y & 4y \end{pmatrix}$, which is fundamental.

To enforce $\det(Y) \neq 0$, extend R by a new variable z and set $\mathfrak{a} = \langle \det(Y) \cdot z - 1 \rangle$. Set $D(z) := z^2 D(\det(Y))$ so that $D(\det(Y) z - 1) = 0 \in \mathfrak{a}$. Then $R[z]/\mathfrak{a}$ is a differential ring.

Let $\mathfrak{m} \subseteq R[z]$ be a differential ideal containing $\det(Y) z - 1$ such that there does not exist any differential ideal $\mathfrak{b} \subseteq R[z]$ with $\mathfrak{m} \subsetneq \mathfrak{b} \subsetneq R[z]$. Such differential ideals $\mathfrak{m}$ are called maximal. They exist because of Hilbert's basis theorem.

Set $\overline{R} := R[z]/\mathfrak{m}$. Then the only differential ideals of $\overline{R}$ are $\{0\}$ and $\overline{R}$, because any nontrivial differential ideal $\{0\} \subsetneq \mathfrak{b} \subsetneq \overline{R}$ containing $p + \mathfrak{a} \in \overline{R}$ with $p \in R[z] \setminus \mathfrak{a}$ would give rise to a differential ideal between $\mathfrak{m}$ and $R[z]$, which is impossible.

This has the following consequences:

(1) $\overline{R}$ is an integral domain.

(2) If $C := \mathrm{Const}\, K$ is algebraically closed, then $\mathrm{Const}(\mathrm{Quot}(\overline{R})) = C$.

Proof.

(1) Let $a \in \overline{R} \setminus \{0\}$. We have to show $\nexists\, b \in \overline{R} \setminus \{0\} : a \cdot b = 0$.

Assume first that a is not nilpotent, i.e. $a^n \neq 0 \;\forall n \in \mathbb{N}$. Consider $I := \{ b \in \overline{R} \mid \exists n \in \mathbb{N} : a^n b = 0 \}$. This is a diff. ideal:

$$b \in I \Rightarrow a^n b = 0 \Rightarrow \underbrace{n a^{n-1} D(a) b + a^n D(b)}_{= a^{n-1}(n D(a) b + a D(b))} = 0$$

$$\Rightarrow n D(a) b + a D(b) \in I \underset{b \in I}{\Rightarrow} a D(b) \in I \Rightarrow D(b) \in I.$$

Since $1 \notin I$ because a is not nilpotent, we have $I = \{0\}$. Hence a is not a zero divisor.

Now suppose a is nilpotent, say $a^n = 0$ for some minimal $n \in \mathbb{N}$. Then $n a^{n-1} D(a) = 0$

$\Rightarrow D(a)$ zero divisor (because $a^{n-1} \neq 0$)

$\Rightarrow D(a)$ nilpotent $\Rightarrow J := \{ a \in \overline{R} \mid \exists n \in \mathbb{N} : a^n = 0 \}$ is a differential ideal. $1 \notin J \Rightarrow J = \{0\}$

$\Rightarrow$ there are no nilpotent elements.

(2) "$\supseteq$" clear. "$\subseteq$" ~~Let~~ Suppose there is some element $a \in \mathrm{Const}(\mathrm{Quot}(\overline{R})) \setminus C$.

Consider $I := \{ b \in \overline{R} \mid a \cdot b \in \overline{R} \}$. Then I is a diff ideal: $b \in I \Rightarrow a \cdot b \in \overline{R}, b \in \overline{R}$

$$\Rightarrow \underbrace{D(a \cdot b)}_{\in \overline{R}} = \underbrace{D(a)}_{=0} b + a \cdot D(b) = a \cdot D(b) \in \overline{R}$$

$$\Rightarrow D(b) \in I.$$

$I \neq \{0\} \Rightarrow I = \overline{R} \Rightarrow 1 \in I \Rightarrow a \in \overline{R}$.

$\Rightarrow a \in \mathrm{Const}\, \overline{R} \setminus C$.

$\Rightarrow \langle a - c \rangle$ is a nonzero diff. ideal for all $c \in C$.

$\Rightarrow 1 \in \langle a - c \rangle$ for all $c \in C$.

Remember: $\overline{R} = K[y_{ij}][z]/m$. Consider $X := V(m) \subseteq \overline{K}^{r^2+1}$. We can regard a as a polynomial map $X \to \overline{K}$.

By a theorem from algebraic geometry, we must have $|\mathrm{im}\, a| < \infty$ or $|\overline{K} \setminus \mathrm{im}\, a| < \infty$.

Since $a - c$ is invertible in $\overline{R}$ for every $c \in C$, we have $\mathrm{im}\, a \cap C = \emptyset$, so $C \subseteq \overline{K} \setminus \mathrm{im}\, a$, so $|\overline{K} \setminus \mathrm{im}\, a| = \infty$, so $|\mathrm{im}\, a| < \infty$.

$\Rightarrow \exists\ P \in K[X] \setminus \{0\}: \quad P(a) = 0$

Say $\quad P = p_0 + p_1 X + \dots + p_d X^d$

with $\quad p_d = 1$ and d minimal.

$$\Rightarrow \quad D(P(a)) = \delta_D(P)(a) + \left(\tfrac{d}{dx}P\right)(a) \cdot \underbrace{D(a)}_{=0} = 0$$

$$= D(p_0) + D(p_1)a + \dots + D(p_{d-1})a^{d-1}$$

By the minimality of d, we must have $\quad D(p_0) = D(p_1) = \dots = D(p_{d-1}) = 0$,

$\Rightarrow$ in fact $\quad \exists\ P \in C[X] \setminus \{0\}: \ P(a) = 0$

$\Rightarrow a \in C$. $\square$

<u>Def</u>: $\overline{R}$ is called a Picard-Vessiot-Ring for A over K, and $E := \mathrm{Quot}(\overline{R})$ is called a Picard-Vessiot-field for A over K.

Two differential rings (R_1, D_1), (R_2, D_2) are called isomorphic (as differential rings) if there is a ring iso. isomorphism $\phi: R_1 \to R_2$ with $\phi \circ D_1 = D_2 \circ \phi$.

Thm: Any two Picard-Vessiot rings for a fixed $A \in K^{r \times r}$ are isomorphic.

Proof:

- Take two PV-Rings $R_1 = K[Y_1]$, $R_2 = K[Y_2]$ where Y_1, Y_2 are fundamental matrices of A.
- Consider $R_3 = K[Y_1, Y_2]$ and a maximal differential ideal $I \subseteq R_3$.
- Then the canonic homomorphisms $\phi_1 : R_1 \to R_3/I$, $\phi_2 : R_2 \to R_3/I$ are injective
- Hence, $\phi_1(Y_1)$ and $\phi_2(Y_2)$ are fundamental matrices for A in R_3/I.
- This implies the existence of some $M \in C^{r \times r}$ with $\phi_1(Y_1) = \phi_2(Y_2) \cdot M$.
- Finally; therefore, $\phi_1(R_1) = \phi_2(R_2)$, and thus $R_1 \simeq R_2$. ⊠