In[1]:= `<< zb.m`

Fast Zeilberger Package by Peter Paule and Markus Schorn (enhanced by Axel Riese) — © RISC Linz — V 3.54 (02/23/05)

In[2]:= `? Gosper`

```
Gosper[ function, range],
uses Gosper's algorithm to find a hypergeometric closed form for the sum
of the function over the range,

Gosper[ function, k], computes the hypergeometric forward anti-difference
of function in k, if it exists,

Gosper[ function, range, degree] or

Gosper[ function, k, degree] use Gosper's algorithm with an
undetermined polynomial of given degree in k multiplied to the function.
```

In[3]:= `Gosper[k k!, {k, 0, n}]`

If 'n' is a natural number, then:

Out[3]= $\left\{\sum_{k=0}^{n} k\, k! == -1 + (1+n)\, n!\right\}$

In[4]:= `Prove[]`

In[5]:= `(* Harmonic numbers are not hypergeometric: *)`

In[6]:= `Gosper[1 / k, {k, 1, n}]`

Out[6]= { }

In[7]:= `Gosper[Pochhammer[a, k] / k!, {k, 0, n}]`

If 'n' is a natural number and $a \neq 0$, then:

Out[7]= $\left\{\sum_{k=0}^{n} \frac{\text{Pochhammer}[a, k]}{k!} == \frac{(a+n)\,\text{Pochhammer}[a, n]}{a\, n!}\right\}$

In[8]:= `Gosper[(-1)^k Binomial[m, k], {k, 0, n}]`

If 'n' is a natural number and $m \neq 0$, then:

Out[8]= $\left\{\sum_{k=0}^{n} (-1)^k\,\text{Binomial}[m, k] == -\frac{(-1)^n\,(-m+n)\,\text{Binomial}[m, n]}{m}\right\}$

In[9]:= `Gosper[Binomial[x + k, k], {k, 0, n}]`

If 'n' is a natural number and $1 + x \neq 0$, then:

Out[9]= $\left\{\sum_{k=0}^{n} \text{Binomial}[k+x, k] == \frac{(1+n+x)\,\text{Binomial}[n+x, n]}{1+x}\right\}$

In[10]:=
```
Gosper[1 / (k^2 + Sqrt[5] k - 1), {k, 0, n}]
```

If ‘n’ is a natural number, then:

Out[10]=
$$\left\{\sum_{k=0}^{n} \frac{1}{-1+\sqrt{5}\,k+k^2} == \frac{(1+n)\,(3+n)\left(100+30\sqrt{5}+98\,n+60\sqrt{5}\,n+36\,n^2+15\sqrt{5}\,n^2+6\,n^3\right)}{30\left(-1+\sqrt{5}\,n+n^2\right)\left(\sqrt{5}+2\,n+\sqrt{5}\,n+n^2\right)\left(3+2\sqrt{5}+4\,n+\sqrt{5}\,n+n^2\right)}\right\}$$

In[11]:=
```
Gosper[(4 k - 1) / (2 k - 1) ^2 16^ (-k) Binomial[2 k, k] ^2, {k, 0, n}]
```

If ‘n’ is a natural number, then:

Out[11]=
$$\left\{\sum_{k=0}^{n} \frac{16^{-k}\,(-1+4\,k)\,\text{Binomial}[2\,k,\,k]^2}{(-1+2\,k)^2} == -16^{-n}\,\text{Binomial}[2\,n,\,n]^2\right\}$$

In[12]:=
```
(* Zeilberger’s algorithm *)
```

In[13]:=
```
? Zb
```

Zb[function, range, n, order],
uses Zeilberger’s algorithm to find a recurrence relation of given order in n
for the sum of the function over the range.

Zb[function, k, n, order],
uses Zeilberger’s algorithm to find a recurrence relation of given order in n
for the function. This recurrence is – up to a telescoping part –
free of k.

In both calls, if the order is of the form {ord1, ord2}, Zb tries to find
a recurrence whose order is between ord1 and ord2. Omitting the order is equivalent to
specifying {0, Infinity}.

In[14]:=
```
rec = Zb[Binomial[n, k], {k, 0, n}, n]
```

If ‘n’ is a natural number, then:

Out[14]= {2 SUM[n] - SUM[1 + n] == 0}

In[15]:=
```
(* note that a HoldForm[...] is wrapped around SUM which
 needs to be released if we want to solve the recurrence above *)
```

In[16]:=
```
FullForm[rec]
```

Out[16]//FullForm=

```
List[Equal[Plus[Times[2, HoldForm[SUM[n]]], Times[-1, HoldForm[SUM[Plus[1, n]]]]], 0]]
```

In[17]:=
```
rec = ReleaseHold[rec /. SUM → g]
```

Out[17]= {2 g[n] - g[1 + n] == 0}

In[18]:=
```
RSolve[rec, g[n], n]
```

Out[18]= $\{\{g[n] \to 2^{-1+n}\, C[1]\}\}$

In[19]:=
```
(* Using Zeilberger’s algorithm to derive
 the three term recurrence for Jacobi polynomials: *)
```

In[20]:= **Zb[$\frac{\text{Pochhammer}[\alpha+1,n]}{n!}\frac{\text{Pochhammer}[-n,k]\,\text{Pochhammer}[n+\alpha+\beta+1,k]}{\text{Pochhammer}[\alpha+1,k]\,k!}\left(\frac{1-x}{2}\right)^k$, {k, 0, n}, n]**

If ‘n’ is a natural number, then:

Out[20]= $\{-2(1+n+\alpha)(1+n+\beta)(4+2n+\alpha+\beta)\,\text{SUM}[n]+$
$(3+2n+\alpha+\beta)\left(8x+12nx+4n^2x+6x\alpha+4nx\alpha+\alpha^2+x\alpha^2+6x\beta+4nx\beta+2x\alpha\beta-\beta^2+x\beta^2\right)$
$\text{SUM}[1+n]-2(2+n)(2+n+\alpha+\beta)(2+2n+\alpha+\beta)\,\text{SUM}[2+n]==0\}$

In[21]:= **Prove[]**

In[22]:= **(* Three term recurrence for Laguerre polynomials: *)**

In[23]:= **Zb[Pochhammer[-n, k] / (k!)^2 x^k, {k, 0, n}, n]**

If ‘n’ is a natural number, then:

Out[23]= $\{(-1-n)\,\text{SUM}[n]+(3+2n-x)\,\text{SUM}[1+n]+(-2-n)\,\text{SUM}[2+n]==0\}$