Binary Numbers

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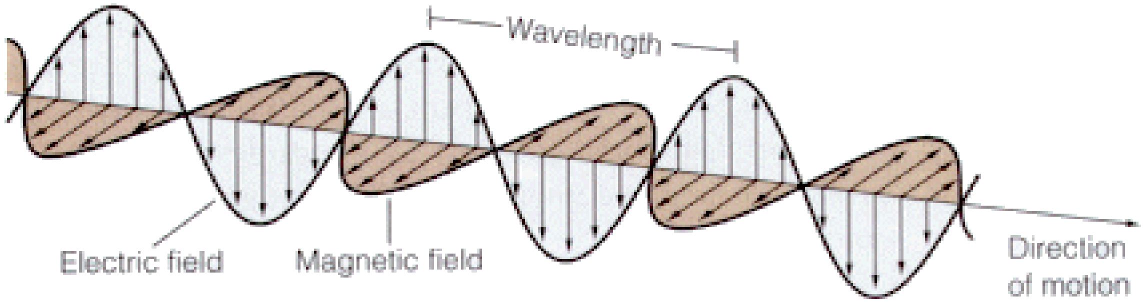
Digital Computers

Todays computers are digital.

- Digital: data are represented by discrete pieces.
  - Pieces are denoted by the natural numbers: 0, 1, 2, 3 . . .
- Analog: data are represented by continuous signals.
  - For instance, electromagnetic waves.



Character Encodings

- ASCII: American Standard Code for Information Interchange.
  - 128 $= 2^7$ characters (letters and other symbols).
  - Also non-printable characters: LF (line-feed).
  - Represented by numbers 0,. . . ,127.

ASCII code	Character
. . .	. . .
10	LineFeed (LF)
. . .	. . .
48–57	0–9
. . .	. . .
65–90	A–Z
. . .	. . .
97–122	a–z
. . .	. . .

Character Encodings

- Text is a sequence of characters.

H	i	,		H	e	a	t	h		e	r	.
72	105	44	32	72	101	97	116	104	101	114	46

- ISO 8859-1 contains $256 = 2^8$ characters (Latin 1).
  - First 128 characters coincide with ASCII standard.
- Unicode contains $65534 = 2^{16} - 2$ characters.
  - First 256 characters coincide with ISO 8859-1 standard.

The ASCII characters are the same in all encodings.

Binary Numbers

In which number system are numbers represented?

- Humans: decimal system.
  - 10 digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
- Computers: binary system.
  - 2 digits: 0, 1.
  - A bit is a binary digit.
  - Physical representation: e.g. high voltage versus low voltage.

Binary numbers are physically easy to represent.

Binary Numbers

- Decimal number 103:

  $$1 * 10^2 + 0 * 10^1 + 3 * 10^0 = 1 * 100 + 0 * 10 + 3 * 1 = 103.$$

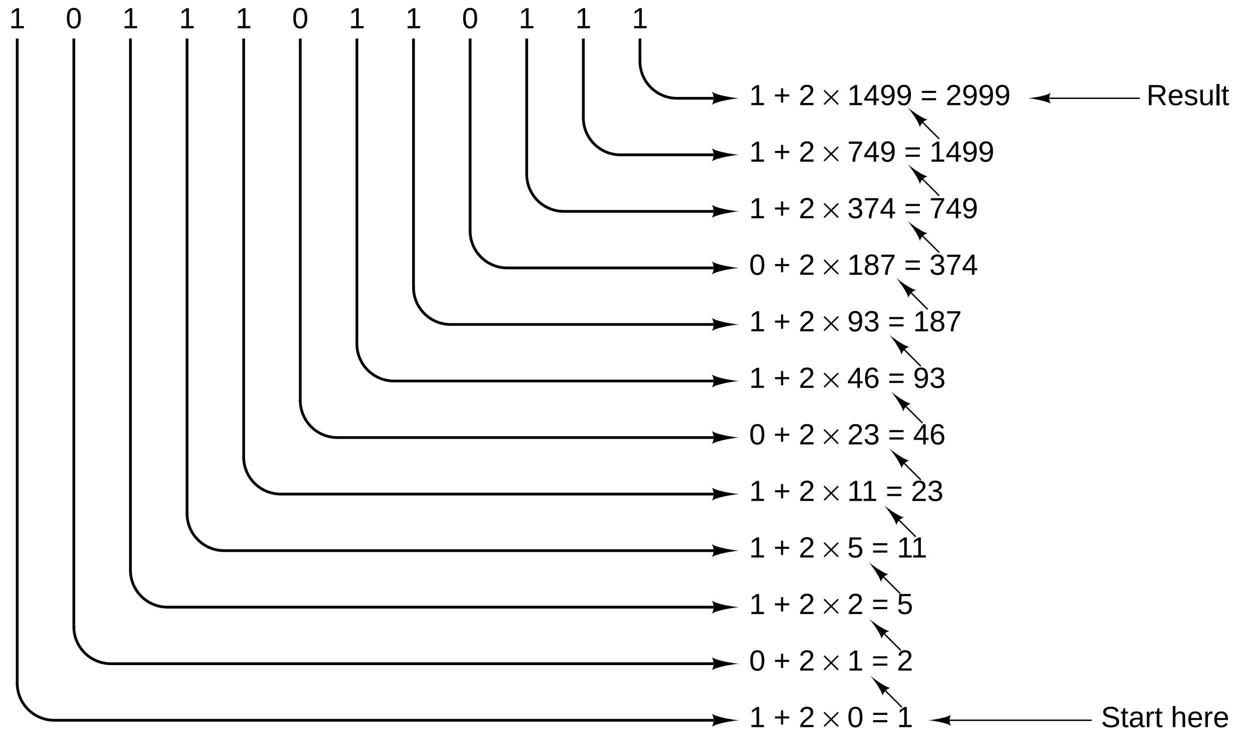
- Binary number 1100111:

  $$1 * 2^6 + 1 * 2^5 + 0 * 2^4 + 0 * 2^3 + 1 * 2^2 + 1 * 2^1 + 1 * 2^0 =$$
  $$1 * 64 + 1 * 32 + 0 * 16 + 0 * 8 + 1 * 4 + 1 * 2 + 1 * 1 = 103$$

- Character g: ASCII code 103.

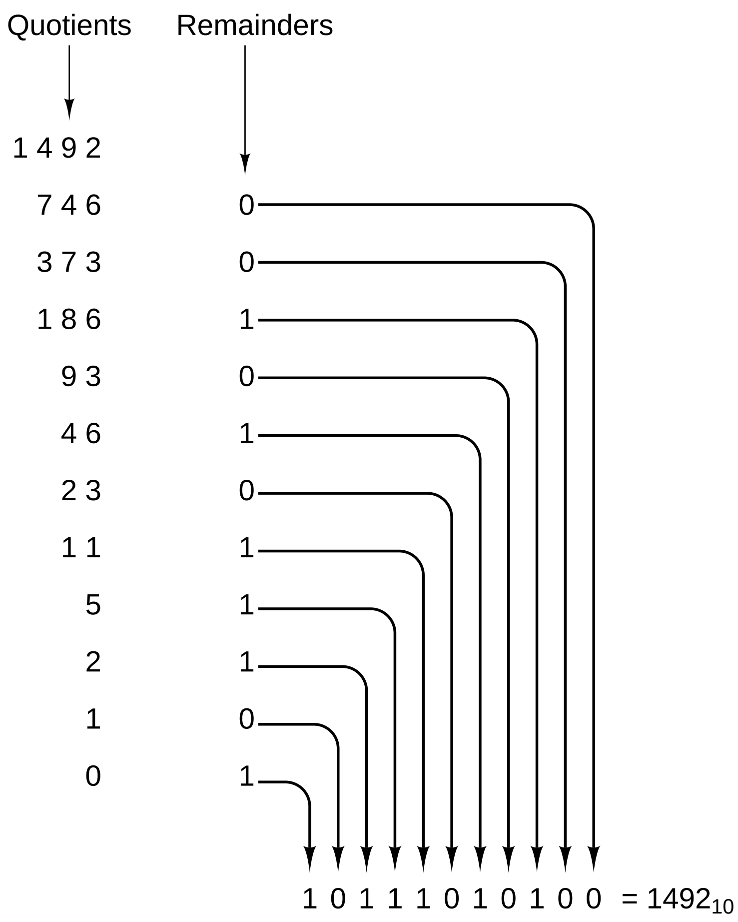
Binary number 1100111 is computer representation of g.

Conversion of Binary to Decimal



Horner's scheme.

Conversion of Decimal to Binary



General Number Systems

- Any base value (radix) b possible.
  - Decimal system: $b = 10$.
  - Binary system: $b = 2$.
- n digits $d_{n-1} \dots d_0$ represent a number m:

  $$m = d_{n-1} * b^{n-1} + \dots + d_0 * b^0 = \Sigma_{0 \leq i < n}\, d_i * b^i.$$
- Number bounds: $0 \leq m < b^n$.
  - Decimal system, 8 digits: $0 \leq m < 10^8 = 100.000.000$.
  - Binary system, 8 digits: $0 \leq m < 2^8 = 256$.
- Example: How many bit does it take to represent a character
  - in ASCII, in the ISO 8859-1 code, in Unicode?
  - $n > \log_b m$.

Other Number Systems

- Octal system:
  - 8 digits 0, 1, 2, 3, 4, 5, 6, 7.
  - One octal digit (6) can be represented by 3 bits (110).
  - Conversion of binary number: 001100111:

001	100	111
1	4	7

- Hexadecimal system:
  - 16 digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F.
  - One hexadecimal digit (B) can be represented by 4 bits (1011).
  - Conversion of binary number 01100111:

0110	0111
6	7

Easy conversion between binary and octal/hexadecimal numbers.

Example Conversions

Example 1

Hexadecimal	1 9 4 8 . B 6
Binary	0 0 0 1 1 0 0 1 0 1 0 0 1 0 0 0 . 1 0 1 1 0 1 1 0 0
Octal	1 4 5 1 0 . 5 5 4

Example 2

Hexadecimal	7 B A 3 . B C 4
Binary	0 1 1 1 1 0 1 1 1 0 1 0 0 0 1 1 . 1 0 1 1 1 1 0 0 0 1 0 0
Octal	7 5 6 4 3 . 5 7 0 4

Hexadecimal/octal numbers are shorter to write.

Unsigned Binary Numbers

Unsigned integers with n bits: from 0 to $2^n - 1$.

Number	Unsigned Integer
000	0
001	1
010	2
011	3
100	4
101	5
110	6
111	7

Computer representation of finite-precision integers.

Signed Binary Numbers

Signed integers with n bits: from -2^{n-1} to $2^{n-1}-1$.

Number	Unsigned Integer	Signed Integer
000	0	+0
001	1	+1
010	2	+2
011	3	+3
100	4	−4
101	5	−3
110	6	−2
111	7	−1

-4	-3	-2	-1	0	1	2	3
100	101	110	111	000	001	010	011

Two's complement representation.

Signed Binary Numbers

Why this representation?

- Arithmetic independent of interpretation.

Binary:	$010 + 101 = 111$
Unsigned:	$2 + 5 = 7$
Signed:	$2 - 3 = -1$

- Computation of representation:
  - Determine representation of -3:
  - Representation of $+3$: 011.
  - Invert representation: 100.
  - Add 1: 101.

Simple implementation in arithmetic hardware.

Binary Arithmetic

Addend	0	0	1	1
Augend	+0	+1	+0	+1
Sum	0	1	1	0
Carry	0	0	0	1

- Overflow:
  - Carry generated by addition of left-most bits is thrown away.
  - Addend and augend are of same sign, result is of opposite sign.

All hardware arithmetic is finite-precision.

Floating-Point Numbers

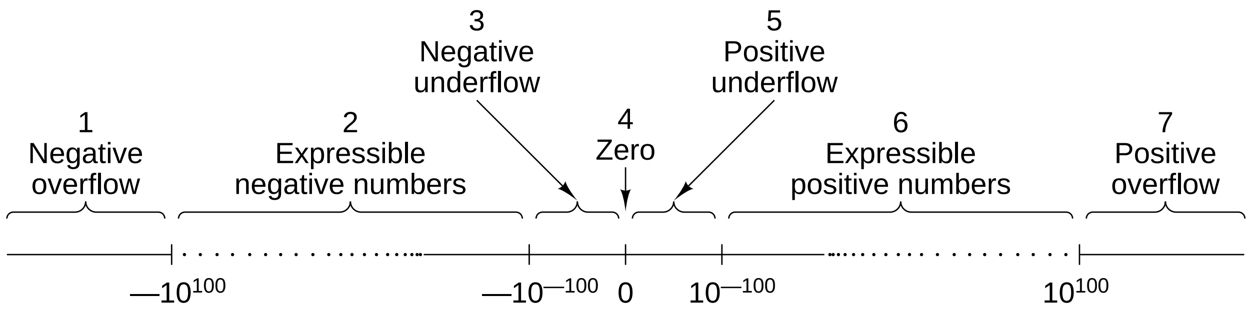
How to represent 1.375?

- Representation: (s, m, e)
  - the sign bit s denotes $+1$ or -1,
  - the mantissa m is a n bit binary number representing the value $m/2^n$ (< 1),
  - the exponent e is a binary number.

  Value: $s * m/2^n * 2^e$
- Example: Eight bit floating point: $0|01011|10$
  - the first bit 0 represents the sign $+1$,
  - the five bit mantissa 01011 represents the fraction $11/32$ (why?),
  - the two bit exponent is 2.

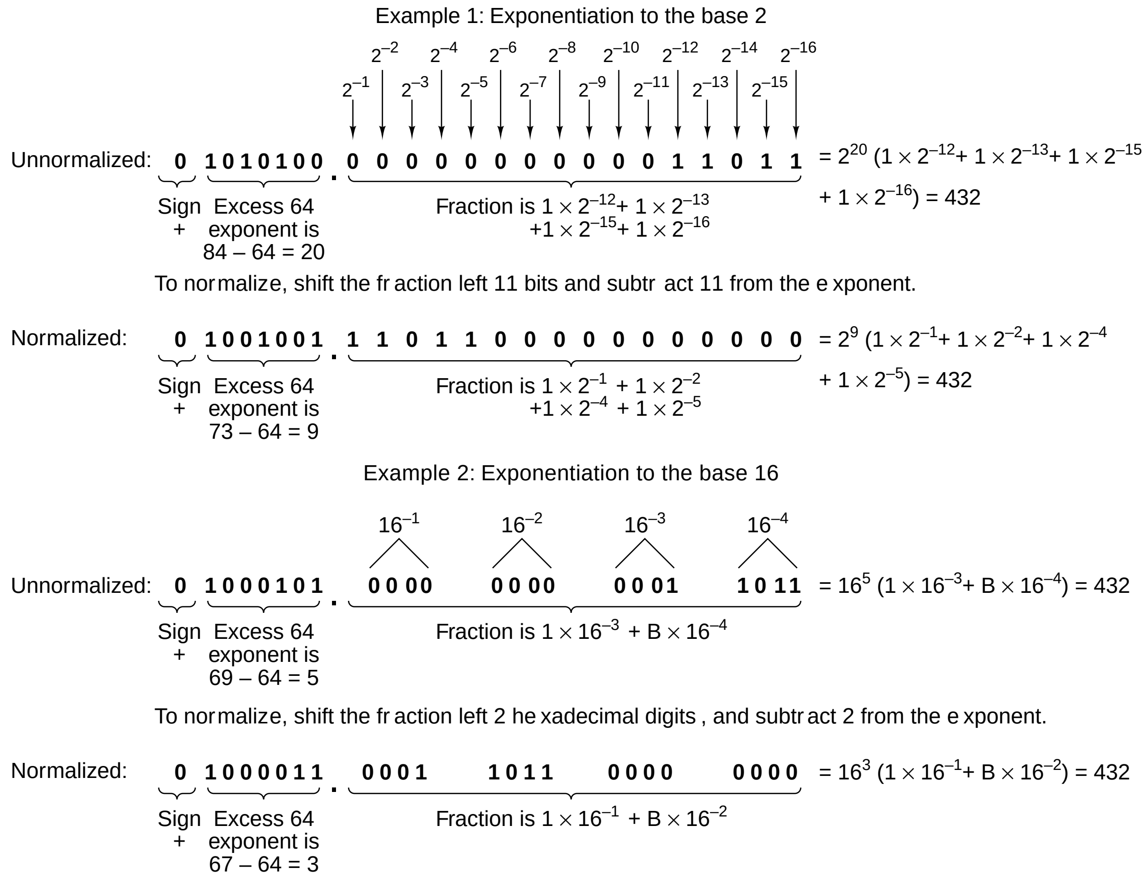
  Value: $+1 * 11/32 * 2^2 = 1.375$

Reals and Floating Points



- Example: fraction with 3 decimal digits, exponent with 2 digits.
  1. Numbers between $-0.999 * 10^{99}$ and $-0.100 * 10^{-99}$.
  2. Zero.
  3. Numbers between $0.100 * 10^{99}$ and $0.999 * 10^{99}$.
- Real values are rounded to the closest floating point value.
  - Overflows and underflows may occur.

Normalized Floating Point Numbers

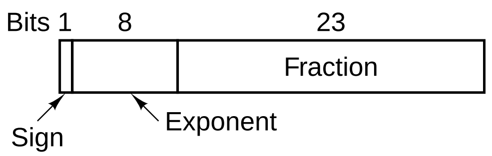


Left-most digit of mantissa is always non-zero.

IEEE Floating-Point Standard 754

IEEE standard for floating point representation (1985).

1. Single precision: 32 bits $(= 1 + 8 + 23)$.
2. Double precision: 64 bits $(= 1 + 11 + 52)$.
3. Extended precision: 80 bits (inside hardware units only).



(a)

(b)

IEEE Floating Point Characteristics

Item	Single Precision	Double Precision
Smallest normalized number	2^{-126}	2^{-1022}
Largest normalized number	2^{128}	2^{1024}
Decimal range	10^{-38} to 10^{38}	10^{-308} to 10^{308}
Smallest denormalized number	10^{-45}	10^{-324}

- Denormalized numbers: first bit of mantissa 0.
  - Distinguished from normalized numbers by 0 exponent.
  - Used to represent very small floating point numbers.
  - Avoid underflows by giving up precision of mantissa.
  - Smallest value: 1 in the rightmost bit, rest 0.

IEEE Numerical Types

Normalized	±	0 < Exp < Max	Any bit pattern
Denormalized	±	0	Any nonzero bit pattern
Zero	±	0	0
Infinity	±	1 1 1…1	0
Not a number	±	1 1 1…1	Any nonzero bit pattern

Sign bit

- Two zeros (positive and negative).
- Plus and minus infinity (overflows).
- NaN (Not a Number = infinity / infinity).