Chain Partition Analysis

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Wos wüst denn du immer

mit deine Köge...?

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MacMahon's Partition Analysis III: The Omega Package

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Polyhedral Omega: a New Algorithm for Solving Linear Diophantine Systems

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Abstract. Polyhedral Omega is a new algorithm for solving linear Diophantine systems (LDS), i.e., for computing a multivariate rational function representation of the set of all non-negative integer solutions to a system of linear equations and inequalities. Polyhedral Omegacombines methods from partition analysis with methods from polyhedral geometry. In particular, we combine MacMahon's iterative approach based on the Omega operator and explicit formulas for its evaluation with geometric tools such as Brion decompositions and Barvinok's short rational function representations. In this way, we connect two recent branches of research that have so far remained separate, unified by the concept of symbolic cones which we introduce.

Chain Partitions

$(\Pi, \preceq)$ — finite graded poset with $\hat{0}$ and $\hat{1}$

$\phi : \Pi \setminus \{\hat{0}, \hat{1}\} \to \mathbb{Z}_{>0}$ order preserving

(Π, ϕ)-chain partition of $n \in \mathbb{Z}_{>0}$: $n = \phi(c_m) + \phi(c_{m-1}) + \cdots + \phi(c_1)$
for some multichain $\hat{1} \succ c_m \succeq c_{m-1} \succeq \cdots \succeq c_1 \succ \hat{0}$

$\mathrm{cp}_{\Pi,\phi}(k) := \#$(chain partitions of k) $\qquad \mathrm{CP}_{\Pi,\phi}(z) := 1 + \sum_{k>0} \mathrm{cp}_{\Pi,\phi}(k)\, z^k$

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Example: $A = \{a_1 < a_2 < \cdots < a_d\} \subset \mathbb{Z}_{>0} \qquad \Pi = [d] \qquad \phi(j) := a_j$

$\longrightarrow$ $\mathrm{cp}_{\Pi,\phi}(k)$ is the restricted partition function with parts in A

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$\operatorname{cp}_{\Pi,\phi}(k) := \#(\text{chain partitions of } k)$ $\qquad \operatorname{CP}_{\Pi,\phi}(z) := 1 + \sum_{k>0} \operatorname{cp}_{\Pi,\phi}(k)\, z^k$

ϕ is ranked if $\operatorname{rank}(a) = \operatorname{rank}(b) \Longrightarrow \phi(a) = \phi(b)$

Π is Eulerian if it is pure and every interval has as many elements of even rank as of odd rank

Theorem $(-1)^{\operatorname{rank}(\Pi)} \operatorname{CP}_{\Pi,\phi}(\frac{1}{z}) \;=\; z^{\sum \text{distinct ranks}} \operatorname{CP}_{\Pi,\phi}(z)$

Eulerian Simplicial Complexes

Γ — simplicial complex (collection of subsets of a finite set, closed under taking subsets)

$f_j \ := \ \#\ (j+1)$-subsets $\ = \ \#$ faces of dimension j

$$h(z) \ := \ \sum_{j=0}^{d+1} f_{j-1}\, z^j\, (1-z)^{d+1-j}$$

Theorem (Everyone 19xy) If Γ is Eulerian then $z^{d+1}\, h(\frac{1}{z}) \ = \ h(z)$.

Key example (Dehn–Sommerville): $\Gamma =$ boundary complex of a simplicial polytope

Chain Partitions for Simplicial Complexes

$\Pi = \Gamma \cup \{\hat{1}\}$ for a d-simplicial complex Γ with ground set V

$\phi(\sigma) = \mathrm{rank}(\sigma) = |\sigma|$

(Π, ϕ)-chain partition of $n \in \mathbb{Z}_{>0}$: $n = \phi(c_m) + \phi(c_{m-1}) + \cdots + \phi(c_1)$
for some multichain $\hat{1} \succ c_m \succeq c_{m-1} \succeq \cdots \succeq c_1 \succ \hat{0}$

$$\mathrm{cp}_{\Pi,\phi}(k) \ := \ \#(\text{chain partitions of } k) \ = \ \sum_{j=0}^{d+1} f_{j-1} \binom{k}{j}$$

Canonical geometric realization of Γ in $\mathbb{R}^V$:

$$\mathcal{R}[\Gamma] \ := \ \left\{ \mathrm{conv}\{\mathbf{e}_v : \, v \in \sigma\} : \, \sigma \in \Gamma \right\}$$

An Ehrhartian Interlude for Polytopal Complexes

$\mathcal{C}$ — d-dimensional complex of lattice polytopes
with Euler characteristic $1 - (-1)^{d+1}$

Ehrhart polynomial $\operatorname{ehr}_{\mathcal{C}}(k) := \# \left(k \, |\mathcal{C}| \cap \mathbb{Z}^d \right)$

We call $\mathcal{C}$ self reciprocal if $\operatorname{ehr}_{\mathcal{C}}(-k) = (-1)^d \operatorname{ehr}_{\mathcal{C}}(k)$. Equivalently,

$\operatorname{Ehr}_{\mathcal{C}}(z) := 1 + \sum_{k>0} \operatorname{ehr}_{\mathcal{C}}(k) \, z^k = \frac{h_{\mathcal{C}}^*(z)}{(1-z)^{d+1}}$ satisfies $z^{d+1} h_{\mathcal{C}}^*(\frac{1}{z}) = h_{\mathcal{C}}^*(z)$

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$$\operatorname{Ehr}_{\mathcal{C}}(z) := 1 + \sum_{k>0} \operatorname{ehr}_{\mathcal{C}}(k) \, z^k = \frac{h^*_{\mathcal{C}}(z)}{(1-z)^{d+1}}$$ satisfies $z^{d+1} h^*_{\mathcal{C}}(\frac{1}{z}) = h^*_{\mathcal{C}}(z)$

Key examples:

$\mathcal{C} =$ boundary complex of a lattice polytope

$\mathcal{C} =$ Eulerian complex of lattice polytopes

Back to Chain Partitions for Simplicial Complexes

$\Pi = \Gamma \cup \{\hat{1}\}$ for a $(d-1)$-simplicial complex Γ with ground set V

$\phi(\sigma) = \mathrm{rank}(\sigma) = |\sigma|$ $\qquad$ $\mathcal{R}[\Gamma] := \left\{ \mathrm{conv}\{\mathbf{e}_v : \, v \in \sigma\} : \, \sigma \in \Gamma \right\}$

(Π, ϕ)-chain partition of $n \in \mathbb{Z}_{>0}$: $\; n = \phi(c_m) + \phi(c_{m-1}) + \cdots + \phi(c_1)$
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Observation 1 $\; \mathrm{cp}_{\Pi,\phi}(k) \; := \; \#(\text{chain partitions of } k) \; = \; \mathrm{ehr}_{\mathcal{R}[\Gamma]}(k)$

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$\Pi = \Gamma \cup \{\hat{1}\}$ for a $(d-1)$-simplicial complex Γ with ground set V

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Observation 1 $\; \mathrm{cp}_{\Pi,\phi}(k) \; := \; \#(\text{chain partitions of } k) \; = \; \mathrm{ehr}_{\mathcal{R}[\Gamma]}(k)$

Observation 2 $\; \mathrm{CP}_{\Pi,\phi}(k) \; = \; \mathrm{Ehr}_{\mathcal{R}[\Gamma]}(z) \; = \; \dfrac{h^*_{\mathcal{R}[\Gamma]}(z)}{(1-z)^{d+1}} \; = \; \dfrac{h_\Gamma(z)}{(1-z)^{d+1}}$

Corollary If Γ is Eulerian then $\; z^{d+1} \, h_\Gamma(\frac{1}{z}) \; = \; h_\Gamma(z)$.