Gröbner fan and universal characteristic sets of prime differential ideals[1]

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Abstract

The concepts of Gröbner fan and universal Gröbner basis have been introduced by Mora and Robbiano as invariants of a polynomial ideal with respect to the choice of a term ordering. In short, the Gröbner fan of an ideal consists of Gröbner cones, each of which corresponds to the set of all term orderings producing the same Gröbner basis. The universal Gröbner basis is a set of polynomials which is a (non-reduced) Gröbner basis of the ideal for any term ordering. These concepts served as a theoretical basis for studying the dependence of the Gröbner basis on the choice of the term ordering, as well as for developing efficient algorithms for transformation of Gröbner bases from one ordering to another (e.g. the Gröbner walk method introduced by Collart, Kalkbrener and Mall).

Characteristic sets of prime differential ideals play a role similar to that of the Gröbner bases of algebraic ideals. In particular, they allow to check the membership of a differential polynomial in a prime differential ideal by comparing the result of reduction of the polynomial with respect to a characteristic set with zero. If the ideal is known to be prime in advance, the algorithm for the computation of the characteristic set is very similar to the Buchberger algorithm. However, unlike Gröbner bases, characteristic sets do not generate the ideal. Moreover, there exist many characteristic sets corresponding to a given ranking, and there is no canonical one among them to play the role of an analogue of a reduced Gröbner basis.

We generalise the concepts of Gröbner fan and universal Gröbner basis to the case of characteristic sets of prime differential ideals, which in this case become invariants of the prime differential ideal with respect to the choice of a ranking on partial derivatives (we restrict ourselves to Riquier rankings). We define the differential Gröbner cone to be the set of all rankings producing characteristic sets of the same rank and show that this set is indeed an open convex cone in an appropriate vector space. We prove that for each cone there exists a set of polynomials which is characteristic for every ranking from this cone; this set is called a strong characteristic set, and an algorithm for its construction is given.

Next, we show that the set of all differential Gröbner cones is finite for any prime differential ideal (this result also allows to prove the termination of the differential analogue of the Gröbner walk algorithm).

A subset of the ideal is called universal characteristic, if it contains a characteristic set of the ideal with respect to any ranking. We show that every prime differential ideal has a finite universal characteristic set and give an algorithm for its construction. The problem of construction of a minimal universal characteristic set is open, but we show how to find such a set and prove its minimality for the particular prime differential ideal generated by $F = \{u_x^2 - 4u,\ u_{xy}v_y - u + 1,\ v_{xx} - u_x\}$ (the example is taken from [Boulier F., Lemaire F., Maza M.M. PARDI! Publication LIFL 2001-01, 2001]). This example also suggests that construction of a universal characteristic set can help solving a system of non-linear PDE's.

Finally, we have observed a strange phenomenon with this example. Our algorithm, implemented in Maple V Release 5 on a Celeron 900MHz under Red Hat Linux 8, computes the universal characteristic set in 22 sec. The algorithm proceeds by calling the Rosenfeld_Gröbner function from the Difalg package for 37 different rankings; each time all previously constructed polynomials are used as the input of Rosenfeld-Gröbner. However, if the Rosenfeld-Gröbner algorithm is called for some of these rankings directly, with the initial set F as the input, it takes much longer (638 sec. for ranking #19) or causes a memory overflow error (for ranking #27 after 1021 sec).

[1]The work has been supported by the Progetto di Interesse Nazionale "Algebra Commutativa e Computazionale" of the Italian "Ministero dell'Istruzione dell'Università della Ricerca Scientifica Tecnologica and by the RFBR grant no. 02-01-01033.

It is too early to draw general conclusions, since the above behavior may be particular for this example or this implementation of Rosenfeld-Gröbner. However, if the above phenomenon turns out to be quite general, it can be explained as follows. Most polynomials from the universal characteristic set participate in several characteristic sets, but the computational cost of obtaining a particular polynomial through the Rosenfeld-Gröbner algorithm depends on the ranking. Quite possibly, in our example, some polynomials that were obtained for rankings #1–26 at a relatively low cost, were then reused for ranking #27. On the contrary, when the Rosenfeld-Gröbner algorithm was called directly for ranking #27, the cost of computing these polynomials might have become much higher. This reasoning also suggests that it might be interesting to investigate a parallel version of our algorithm, which computes characteristic sets for several rankings simultaneously and shares the obtained polynomials among the processes.