# Algorithmic Combinatorial Spencer Complex

Eduardo Sáenz de Cabezón [*]

## Introduction

In the formal theory of partial differential equations a central issue is the Spencer cohomology [4], which strongly relates the formal theory with commutative algebra and provides a better understanding of the concept of involution [3].

We present a combinatorial complex which cohomology groups are isomorphic to the Spencer cohomology groups of monomial symbols. The combinatorial properties of this complex provide an efficient algorithm which allows high dimensional computations of Spencer cohomology of linear systems of monomial PDE's.

## 1 The Combinatorial Spencer Complex.

For every $n \in \mathbb{N}$ and $q \in \mathbb{N}$ we consider the following sets:

$S^n_q = \{[i_1 \dots i_q] |\, 1 \leq i_j \leq n\}$, $n$-Multi Indices of order $q$, and $\Lambda^n_q = \{[i_1 \dots i_q] |\, 1 \leq i_1 < \dots < i_q \leq n\}$, *ascending* $n$-Multi Indices of order $q$.

Beginning with a given $E^n_q \subseteq S^n_q$, we generate a sequence $\{E^n_r\}_{r \geq q}$ such that $E^n_r \subseteq S^n_r \, \forall r > q$ given by

$$E^n_r = \{[I \cup i] \text{ s.th. } [I] \in E^n_{r-1}\,, 1 \leq i \leq n\}$$

and the set of complementary subsets $\{M^n_r\}_{r \geq q}$, where $M^n_r = S^n_r - E^n_r$. If we identify the set $E^n_q$ with a system of monomial partial differential equations, then $M^n_q$ can be identified with its symbol, and the corresponding sequences represent their prolongations.

Given a field $\mathbf{k}$, we denote

$$\mathbf{k}S^n_r \otimes \Lambda^n_s = \{\alpha[I|\,\gamma] \text{ s.th. } \alpha \in \mathbf{k}\,, [I] \in S^n_r\,, [\gamma] \in \Lambda^n_s\},$$

we define the following homomorphism:

$$\delta : \mathbf{k}S^n_r \otimes \Lambda^n_s \longrightarrow \mathbf{k}S^n_{r-1} \otimes \Lambda^n_{s+1}$$

$$\alpha[I|\,\gamma] \mapsto \sum_{j=1}^{r} sgn([i_j \cup \gamma])\alpha[I - \{i_j\}|\, sort(i_j \cup \gamma)]$$

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and the following sequences

$$\mathcal{S}_{r+s}(S^n_q) : \cdots \to S^n_r \otimes \Lambda^n_s \xrightarrow{\delta} S^n_{r-1} \otimes \Lambda_{s+1} \to \cdots$$

these are exact sequences that can be restricted to complexes

$$\mathcal{S}_{r+s}(M^n_q) : \cdots \to M^n_r \otimes \Lambda^n_s \xrightarrow{\delta} M^n_{r-1} \otimes \Lambda^n_{s+1} \to \cdots$$

where $M_r = S_r \, \forall r < q$, which cohomology groups are the *Combinatorial Spencer Cohomology Groups* of $M^n_q$ or $E^n_q$, i.e. $H^{r,s}(M^n_q)$ (resp. $H^{r,s}(E^n_q)$).

An important role will be played by critical indices and their properties:

**Definition 1.1.** Let $[I] \in M^n_r$, then if $[I \cup \{i\}] \in E^n_{r+1}$ we say that $i \in \{1, \ldots n\}$ is a *critical index* for $[I]$. We denote $N_1([I])$ the set of critical indices for $[I]$; $N_k([I]) = \{[j_1 \ldots j_k] \in \Lambda^n_k | j_i \in N_1([I]) \forall i \in \{1, \ldots, k\}\}$.

If $[I] = [I' \cup \{i\}]$ ; $[I] \in M^n_r$, $[I'] \in M^n_{r-1}$ we say that $[I']$ is an *ancestor* of $[I]$, and that $[I]$ is a *descendant* of $[I']$. The set of ancestors of $[I]$ will be denoted $\mathcal{A}([I])$.

**Lemma 1.2.** *For all* $[I] \in M^n_r$,

$$N_1([I]) = \bigcup_{I' \in \mathcal{A}([I])} N_1([I'])$$

**Lemma 1.3.** *Let* $[I] \in M^n_r$ , $[\gamma] = [j_1 \ldots j_s] \in \Lambda^n_s$, *then*

$$[I|\gamma] \in Im(\delta) \iff [\gamma] \notin N_s([I])$$

**Proposition 1.4.** *Let* $[I] \in M^n_r$, $[\gamma] \in N_s([I])$; *if* $\exists I' \in \mathcal{A}([I])$ *such that* $[\gamma] \in N_s([I'])$ *then* $[I|\gamma]$ *is not a summand of a generator of* $H^{r,s}(\mathcal{S})$

## 2 The Algorithm

After making use of the preceding properties to discard elements which will not be present in the summands of generators of cohomology groups, our algorithm divides the remaining in blocks: we say that $[i_1, \ldots, i_r | j_1, \ldots, j_s] \sim [i'_1, \ldots, i'_r | j'_1, \ldots, j'_s] \Leftrightarrow [i_1, \ldots, i_r, j_1, \ldots, j_s] = [i'_1, \ldots, i'_r, j'_1, \ldots, j_s] \in S^n_{r+s}$ (after the corresponding reordering). This is an equivalence relation in $M^n_r \otimes \Lambda^n_s$ and the equivalence classes are the blocks. Every generator of a cohomology group is formed by summands from the same block. The initial blocks can still be reduced with a simple procedure before we apply the final linear algebra step. From the reduced blocks we obtain small linear systems which kernels provide generators of $H^{r,s}(\mathcal{S})$.

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$Algorithm : CSC - Cohomology$

$Input : Set\, E_q^n \subseteq S_q^n$
$Output$ : Cohomology of the Combinatorial Spencer Complex $\mathcal{S}(E_q^n)$

$E \leftarrow E_q^n$; $M \leftarrow S_{q-1}^n$; $r = q$
$\mathcal{I} \leftarrow$ Indices of $M$ :   $\mathcal{I}_{[I]} = N_1([I])$
$\mathcal{P} \leftarrow$ Products of $M$ :   $\mathcal{P}_{[I]} = \bigcup_{k \in \{1 \ldots d\}}$ k-subsets of $N_1(I)$
while not $(Involutive(M))$ do
  $M \leftarrow S_r - E$
  $E \leftarrow E_{r+1}$
  $\mathcal{I}_{[I]} = \bigcup_{I' \in \mathcal{A}(I)} \mathcal{I}_{[I']}$
  $Aux \leftarrow \mathcal{P}$
  $\mathcal{P} \leftarrow$ Products of $M$
    $\mathcal{G} \leftarrow \mathcal{G}_{[I]} = \mathcal{P}_{[I]} - \bigcup_{I' \in \mathcal{A}([I])} Aux_{[I]}$
    $\mathcal{B} \leftarrow$ Make blocks:
      $\mathcal{B}_{[I|\gamma]} = \{[J|\eta],\ J \in M,\ \eta \in \mathcal{P}_{[J]} | [J \cup \eta] = [I \cup \gamma] \in S_{n+s}^n\}$
    while not $(Reduced\,(\mathcal{B}))$ do
      $B_{[I|\gamma]} \leftarrow,$ ReduceBlock($B_{[I|\gamma]}$)
    end while
  Compute blocks $\rightarrow H^{r,s} \forall s$
  $r = r + 1$
end while
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# 3 Conclusions and further work

The presented algorithm makes use of the particular combinatorial properties of symbols of linear systems of monomial PDE's to compute their Spencer cohomology. This is done by reducing computations to solving a number of small linear systems instead of the huge systems that outcome from the brute-force linear algebra approach used before. Comparing the results and amount of work done by these two approaches, it is clear the strength of the new algorithm (see Table 1).

Further work involves continue exploiting the combinatorial properties of the Combinatorial Spencer Complex in order to characterize involutivity of a linear system of monomial equations in terms of the *critical indices*. A second direction is the relation betwen the computation of Spencer cohomology and the completion to involution algorithms given in [1], [2].

Table 1: Number and size of blocks (linear systems) for some examples.

| **n** | **q** | **Max. size of blocks** | **Time elapsed** | **Max. size using brute-force approach** |
|---|---|---|---|---|
| 3 | 3 | 0* | 0'02 s. | 21 × 21 |
| 3 | 4 | 0* | 0'03 s. | 39 × 39 |
| 4 | 3 | 0* | 0'09 s. | 156 × 108 |
| 4 | 4 | 0* | 0'23 s. | 366 × 264 |
| 6 | 3 | 8×13 | 5'268 s. | 1000 × 1380 |
| 6 | 4 | 8 × 13 | 28'761 s. | 4340 × 5190 |

## References

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UNIVERSIDAD DE LA RIOJA (Spain)
Departamento de Matemáticas y Computación, Edificio Vives,
eduardo.saenz-de-cabezon@dmc.unirioja.es