Equilibrium points of dynamical systems and polynomial resultants

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One of the keys of the qualitative theory of differential equations is the determination of equilibrium points and their properties of stability. They are the basic pieces to depict an skeleton of the global phase flow of the system from which it is possible to deduce periodic, quasiperiodic or asymptotic behaviors. In particular, we are interested in the location of periodic orbits for perturbed Keplerian systems. They can be obtained, in first approximation, as the equilibria of a one degree of freedom Hamiltonian system resulting after normalization [1]. In this way, we arrive to a system of differential equations of the form

$$\begin{aligned}\dot{g} &= R_1(G,\sigma)\sin 2g,\\ \dot{G} &= R_2(G,\sigma) + R_3(G,\sigma)\cos 2g,\end{aligned}$$

where $\sigma \in \mathbf{R}^k$ is a vector of real parameters and R_1, R_2, R_3 are rational funtions of G. Moreover, excluding the meaningless case $G = 0$, the critical points are the solutions of the nonlinear system

$$\begin{aligned}&P_1(G,\sigma)\sin 2g = 0,\\ &P_2(G,\sigma) + P_3(G,\sigma)\cos 2g = 0,\end{aligned}$$

such that $g \in [0, 2\pi)$, $G \in [\alpha, \beta]$ and P_1, P_2, P_3 are real polynomials in G whose coefficients are polynomial functions of the parameters, being α and β two of the parameters.

It is clear that for such a system, critical points appear for $\sin 2g = 0$ and G a root of the polynomial equation

$$\mathcal{P}(G) \equiv P_2(G,\sigma) \pm P_3(G,\sigma) = 0$$

belonging to the interval $[\alpha, \beta]$.

However, the parametric dependence of the coefficients of the polynomial $\mathcal{P}$ and the degree of it (greater than five) prevent for explicit expressions, in terms of the parameters, of the critical points. In this situation, continuation techniques are introduced to obtain a family of equilibrium points as a function of one single parameter while the rest are fixed [2]. This procedure allows one to detect branch points in the parameter space, those points where different families of equilibria collide.

As an alternative to continuation methods, we can use an analytical process not to determine the coordinates of the critical points but the set of branch points in the parameter space (the so called bifurcation lines). Now, branch points are obtained when the number of roots in the interval $[\alpha, \beta]$ changes. In this way, Sturm sequences can be used as it is done in [3]. However, we propose a procedure to find the set of branch points based on the resultant (discriminant) of a polynomial ($\mathrm{Res}(\mathcal{P})$) and the following observation:

The number of roots in the interval $[\alpha, \beta]$ changes if

- One of them reaches the extrema of the interval. Then $\mathcal{P}(\alpha) = 0$, $\mathcal{P}(\beta) = 0$ belong to the set of branch points.
- Two (or more) roots give rise to a multiple root in $[\alpha, \beta]$. Then $\mathrm{Res}(\mathcal{P}) = 0$ and the set of branch points can have non empty intersection.

The most tricky step of the procedure involves the determination of that part of the resultant, if any, made up of branch points. Nevertheless, a multiple root outside the given interval causes the resultant to vanish. To this end, numerical methods must be used to obtain, on the one hand, the intersection points of $\mathcal{P}(\alpha) = 0$, $\mathcal{P}(\beta) = 0$ and $\mathrm{Res}(\mathcal{P}) = 0$ and, on the other hand, the roots of the polynomial $\mathcal{P}$.

We apply this procedure to the problem of finding the frozen periodic orbits of a satellite around an oblate planet, a problem treated only numerically because of its complexity since it depends on three parameters [4]. We obtain, by the first time, the set of branch points and give the global phase portrait of the system as well as the conditions where those configurations, described numerically, persist.

References

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