Planar maps: bijections and applications

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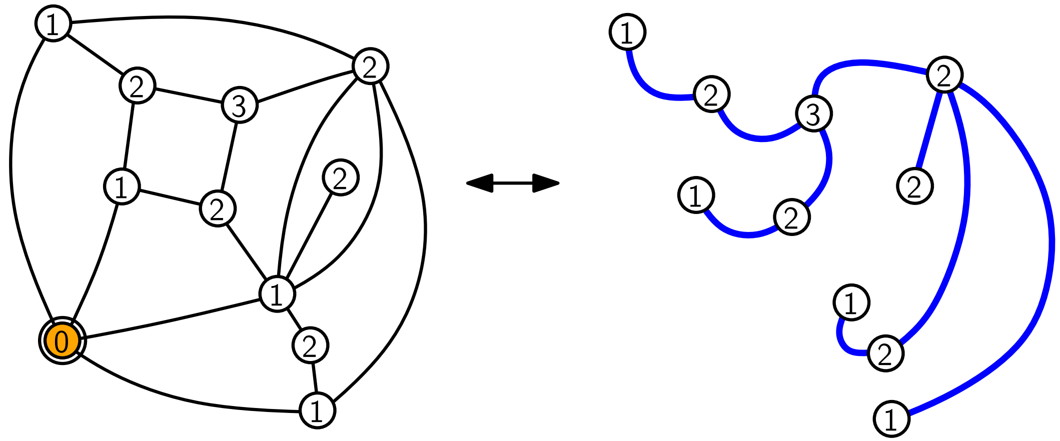
AEC summer school, Hagenberg, 2018

Overview of the course

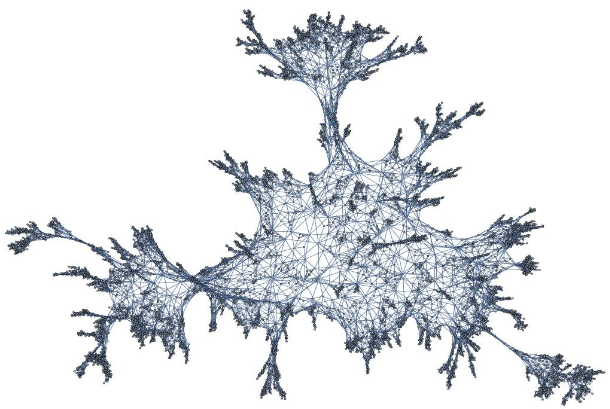
- Planar graphs and planar maps

  - structural aspects

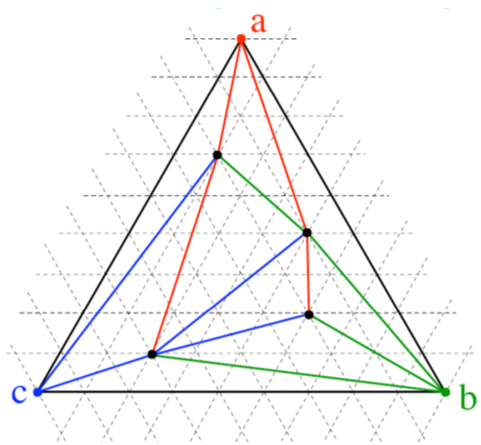
  - enumerative aspects

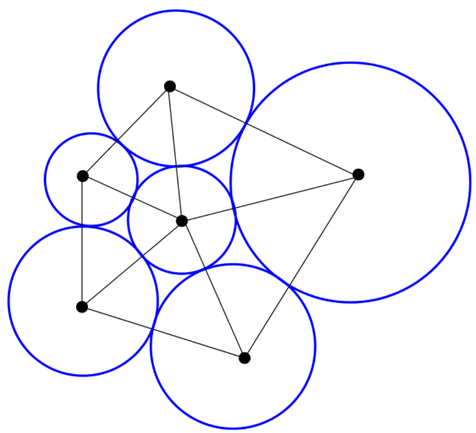


- distances in random maps



- geometric representations



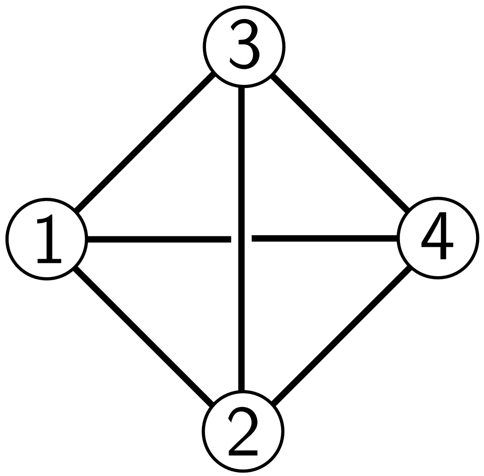


Structural aspects of planar graphs and maps

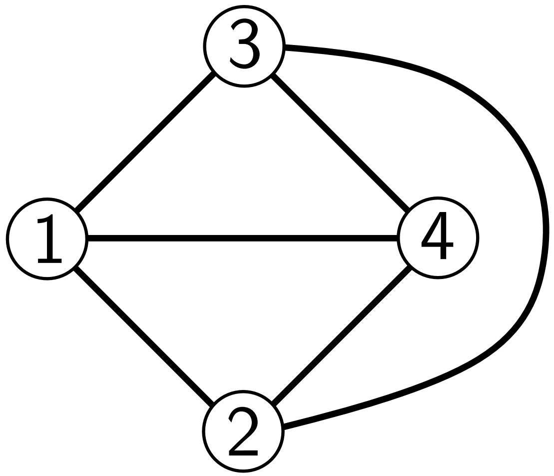
Planar graphs

A graph is called **planar** if it can be drawn crossing-free in the plane

K_4 is planar

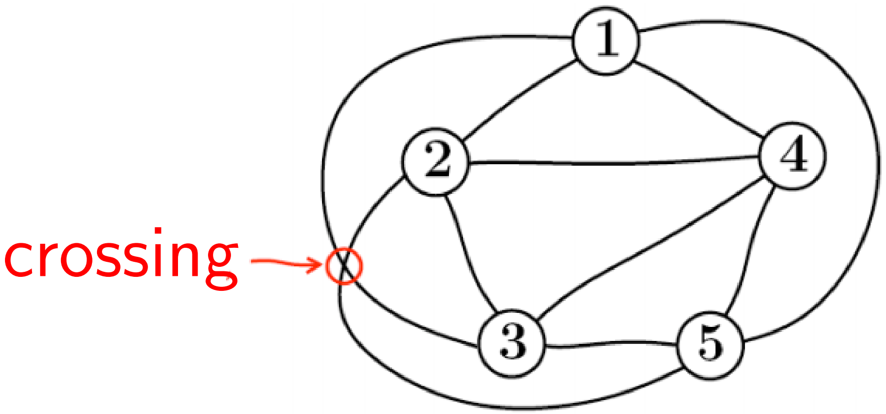


non-planar drawing



planar drawing

K_5 is not planar



(whatever drawing, there is always a crossing)

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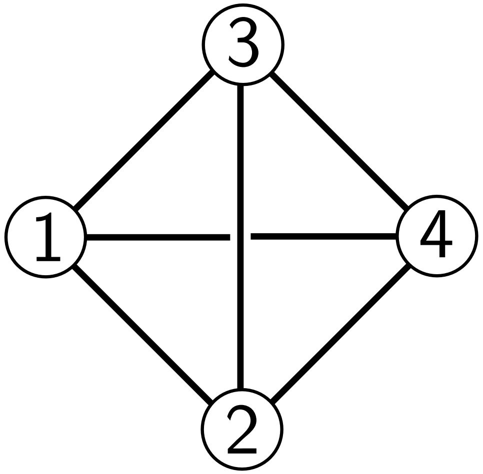
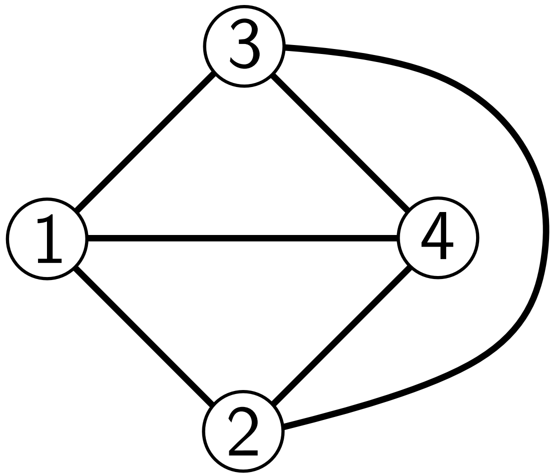


non-planar drawing



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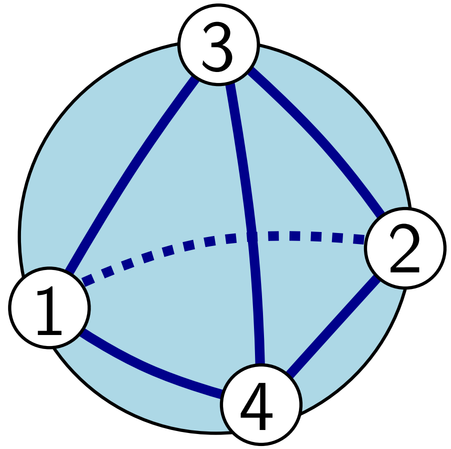
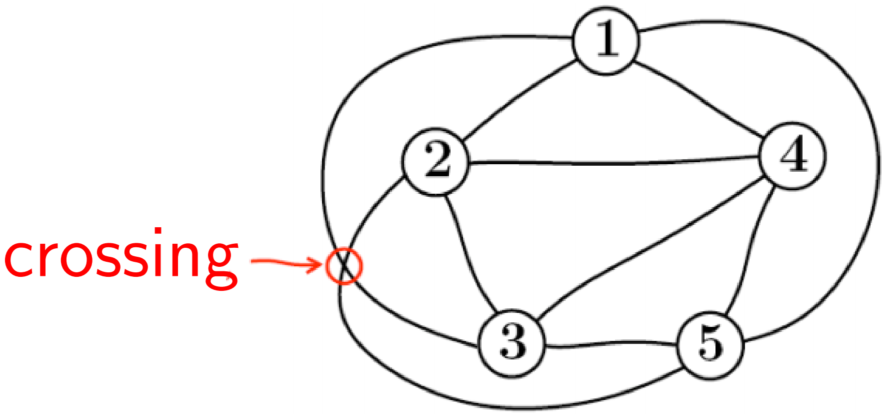


on the sphere

K_5 is not planar



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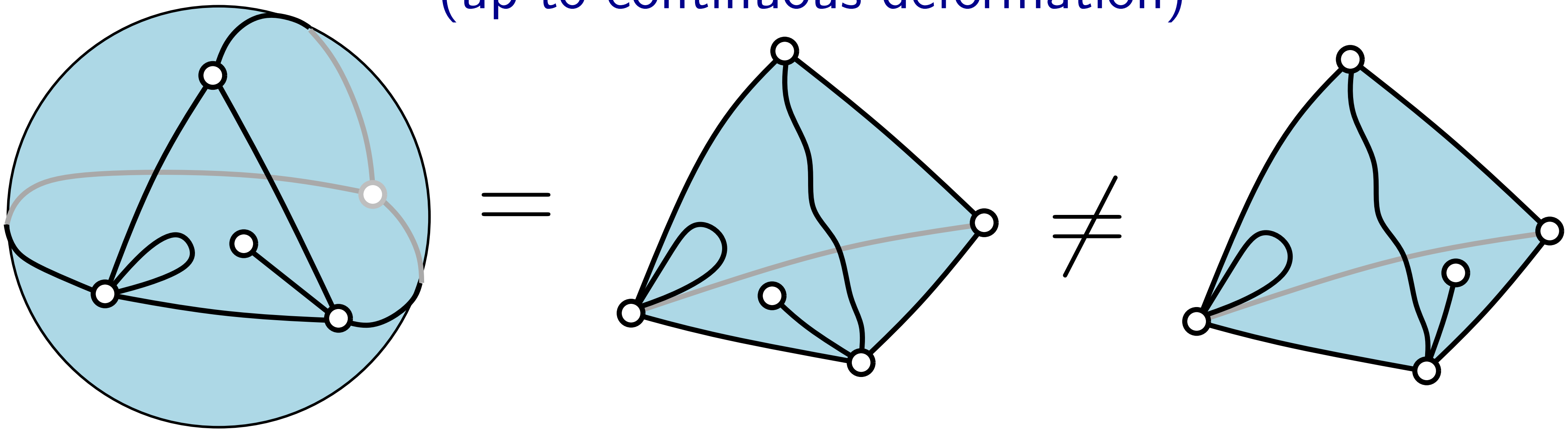
Rk: planar $\leftrightarrow$ can be drawn crossing-free on the sphere

Planar maps

Def. Planar map = connected multigraph embedded on the sphere
(up to continuous deformation)



Rk: a planar graph can have several embeddings on the sphere

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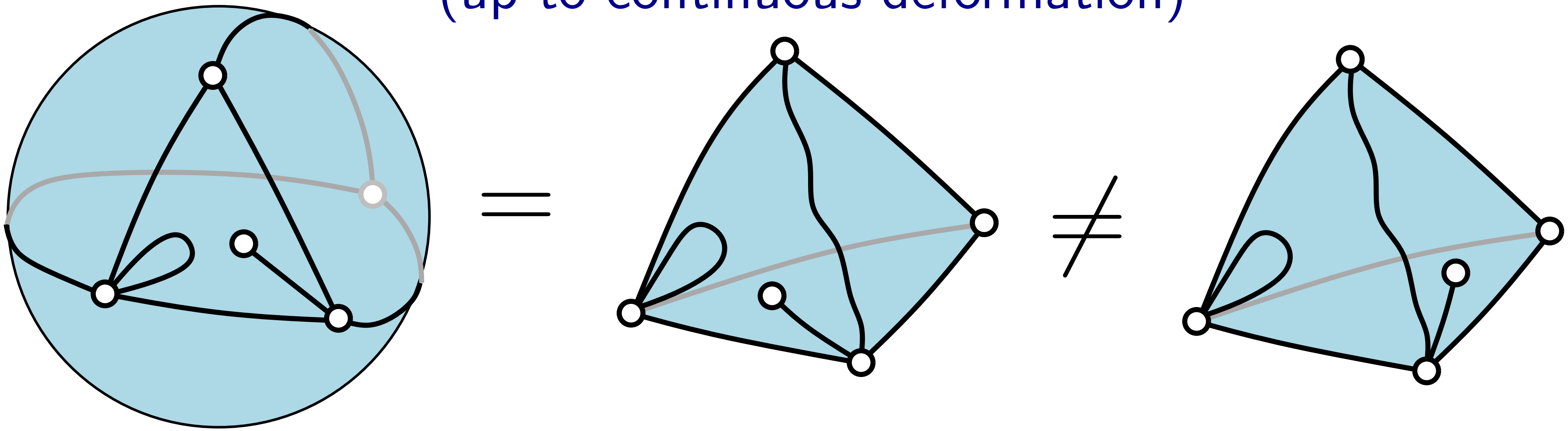
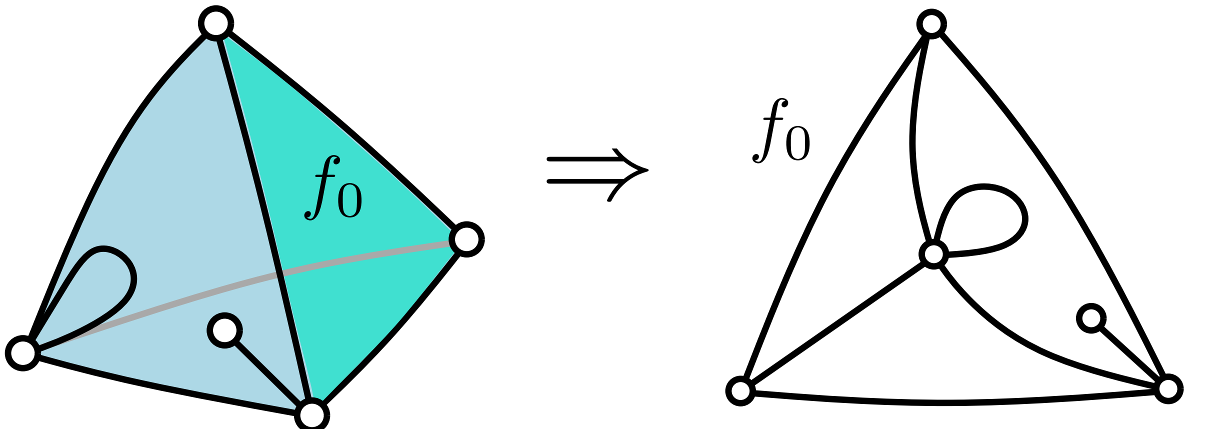


Rk: a planar graph can have several embeddings on the sphere

A map is easier to draw in the plane (implicit choice of an outer face f_0)



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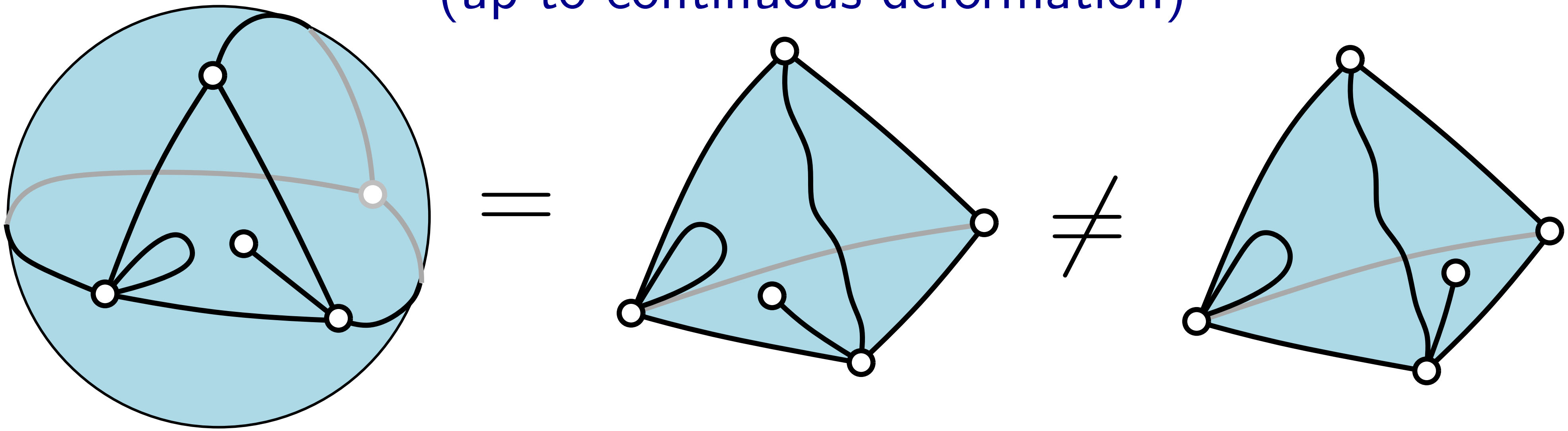

Rk: a planar graph can have several embeddings on the sphere
a map has vertices, edges, and faces

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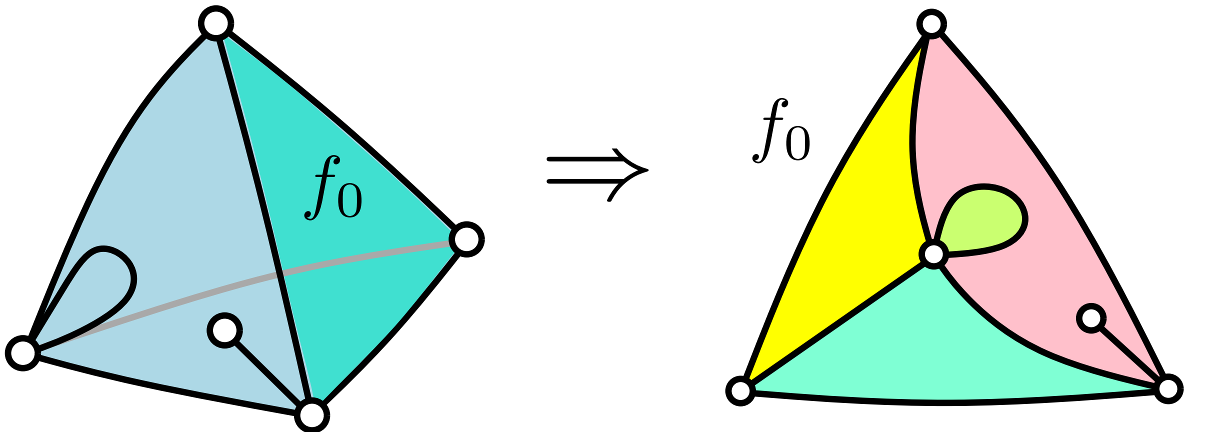
5 faces (including outer one)



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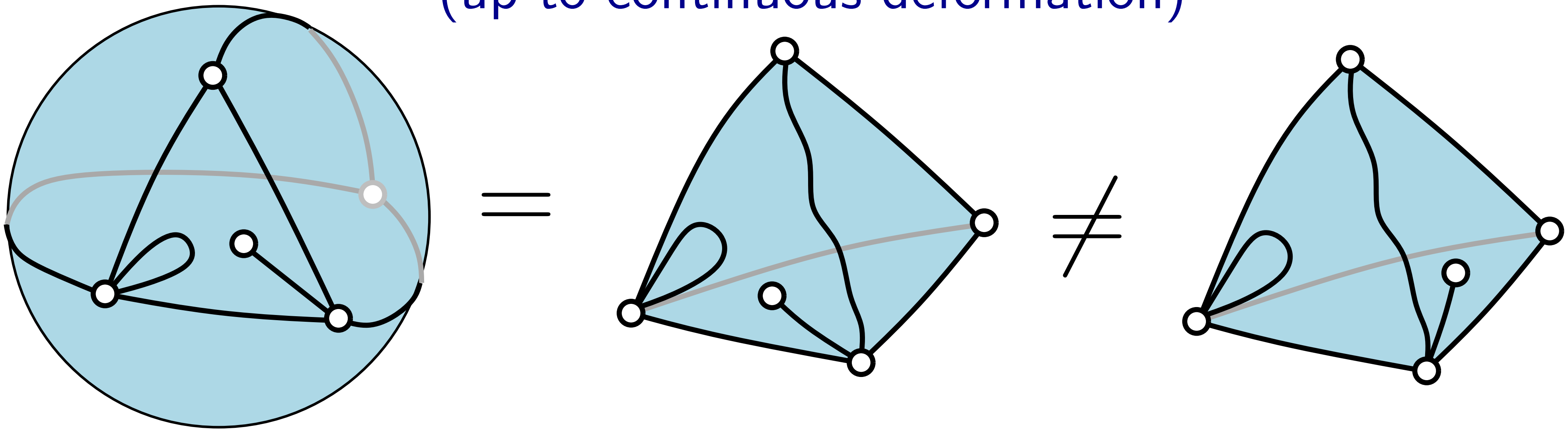

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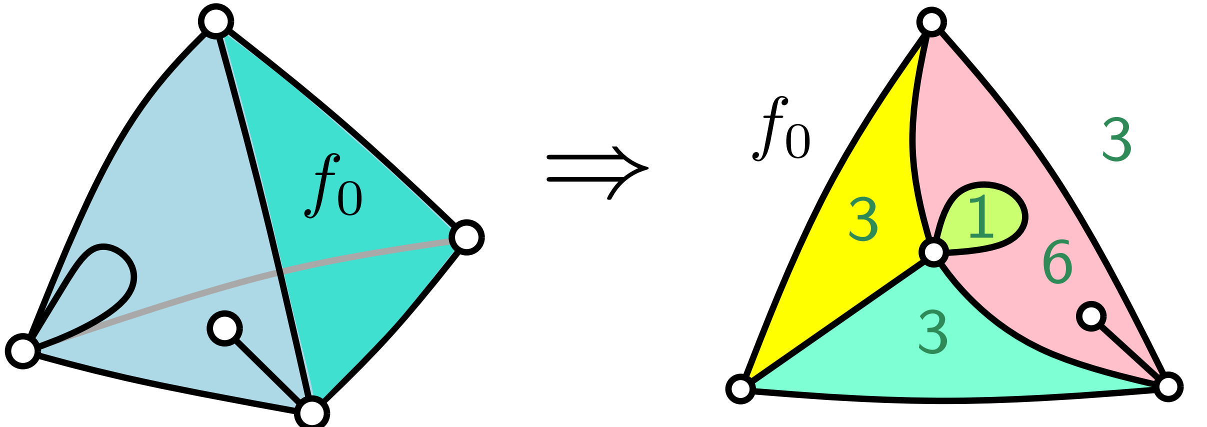
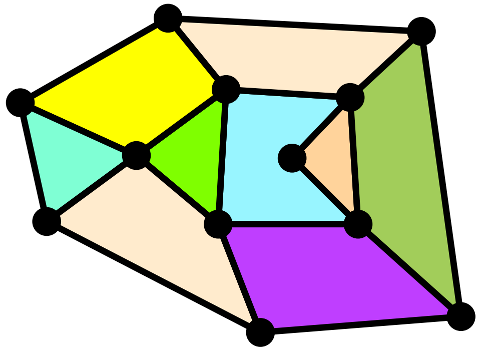
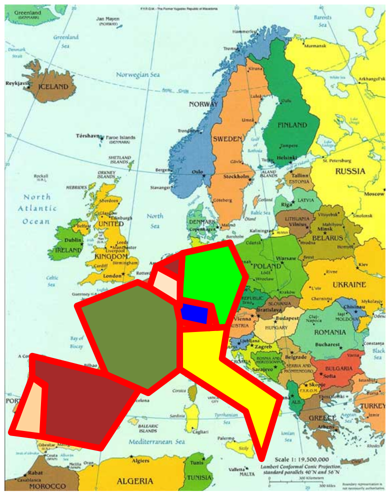


5 faces (including outer one)

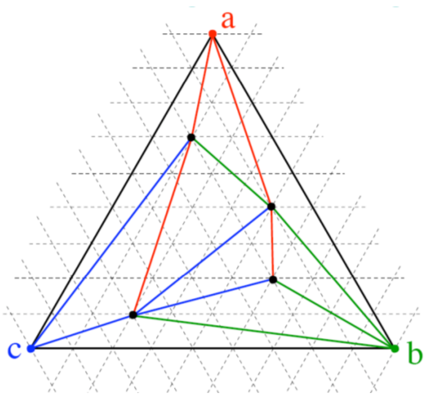
degree of a face
= length of walk around f

Some contexts where maps appear

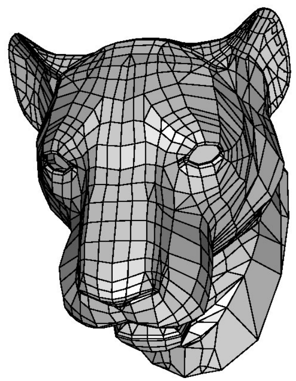




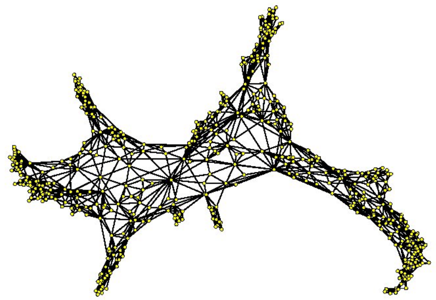
geographic maps
(topological info.)



graph drawing



meshes (CAO)
(combinatorial incidences)

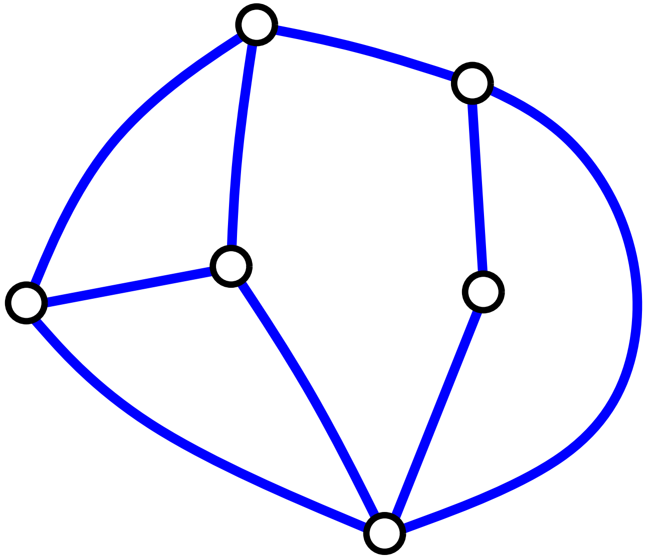


Probability and physics
(random lattices, random surfaces)

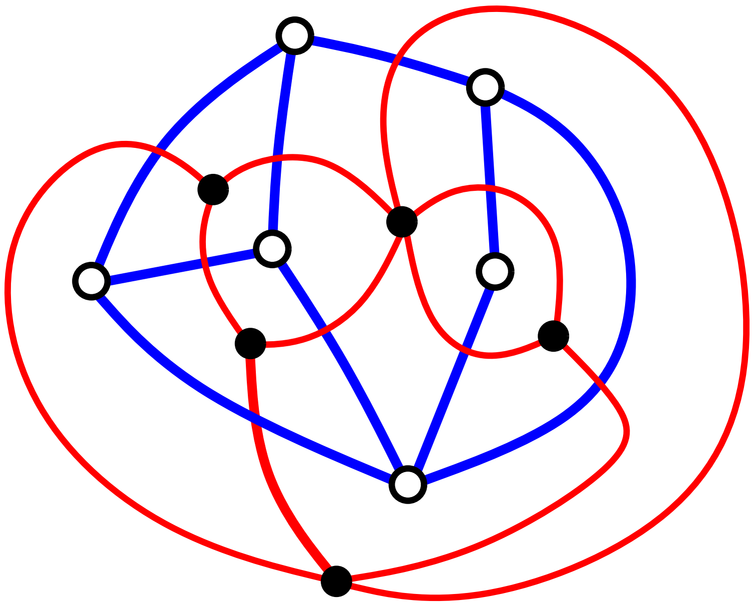
(and also: ramified coverings, factorizations in the symmetric group, classification of surfaces)

Duality for planar maps

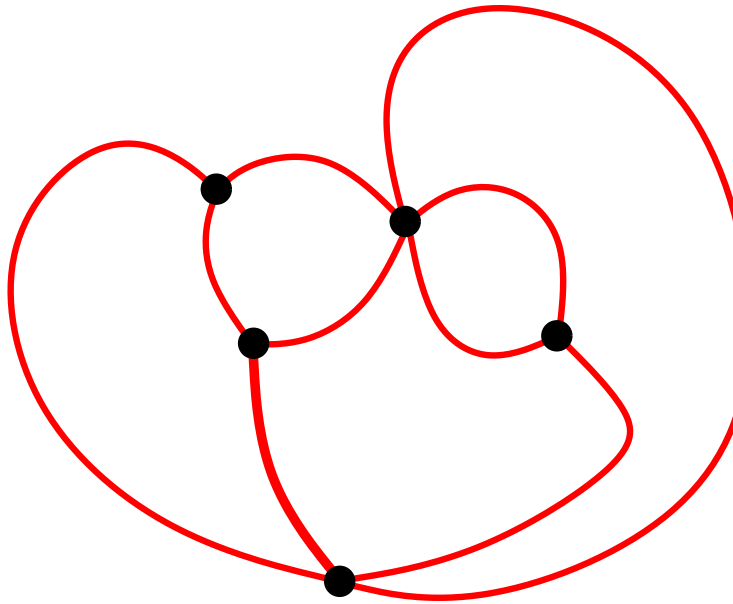
6 vertices, 9 edges, 5 faces



a planar map



the dual map



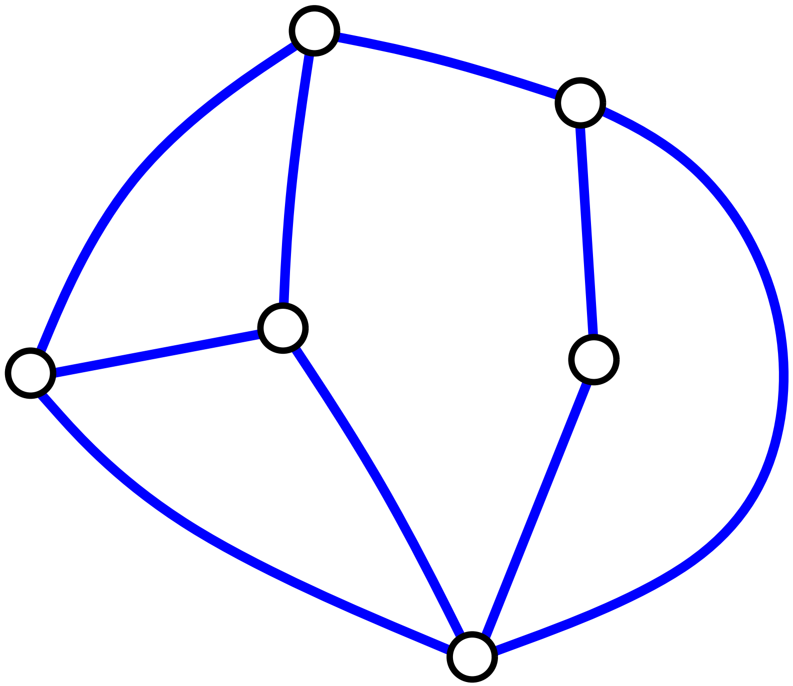
5 vertices, 9 edges, 6 faces

preserves #(edges), **exchanges #(vertices) and #(faces)**

The Euler relation

Let $M = (V, E, F)$ be a planar map. Then

$$|E| = |V| + |F| - 2$$



$$|V| = 6, |E| = 9, |F| = 5$$

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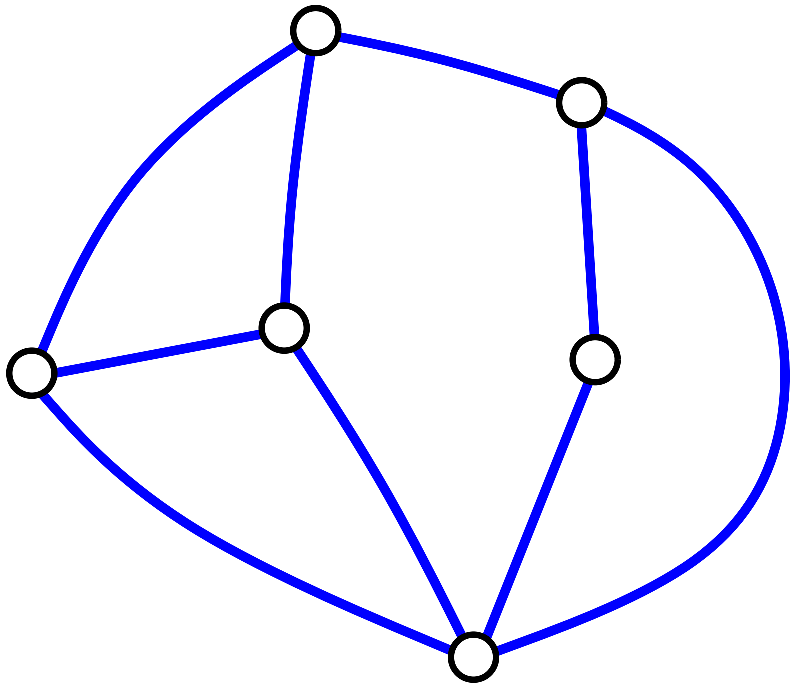
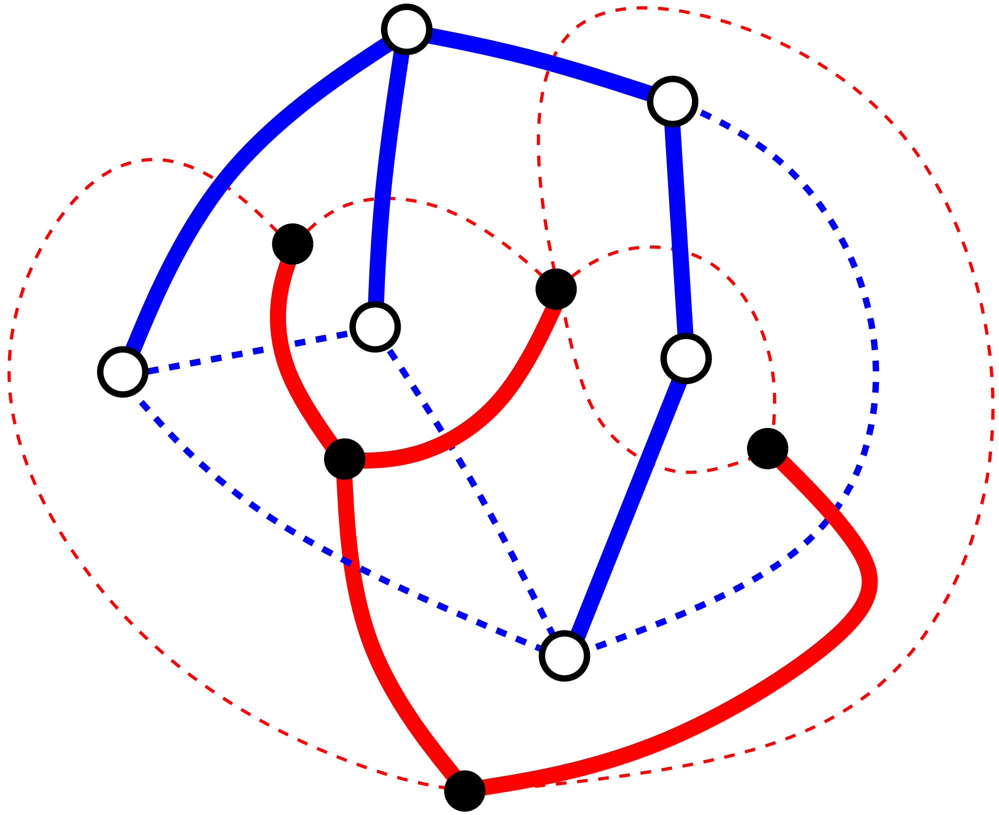


$|V| = 6, |E| = 9, |F| = 5$

Proof using spanning trees

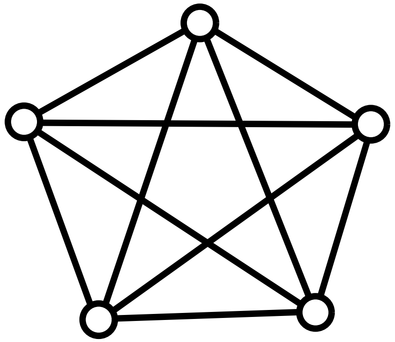
$$|E| = (|V| - 1) + (|F| - 1)$$



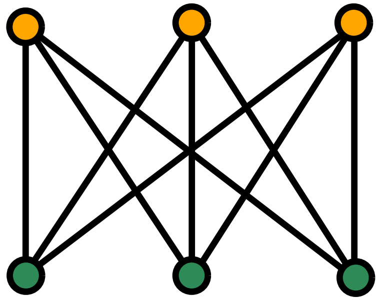
Kuratowski's theorem for planar graphs

The Euler relation implies (exercise!) that K_5 and $K_{3,3}$ are not planar

K_5



$K_{3,3}$



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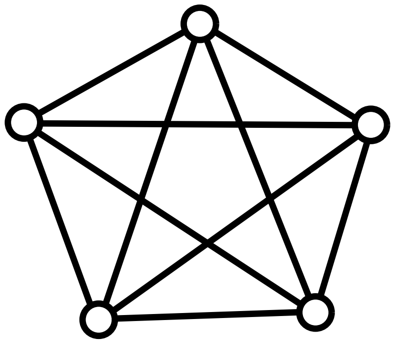

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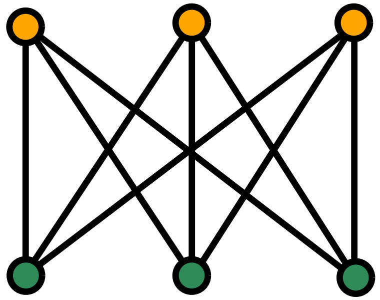
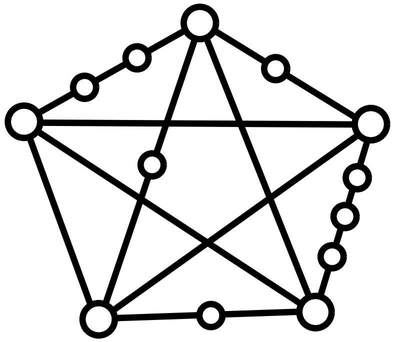


Hence any subdivision of K_5 or $K_{3,3}$ is not planar either



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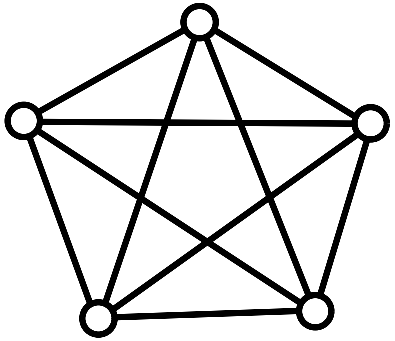

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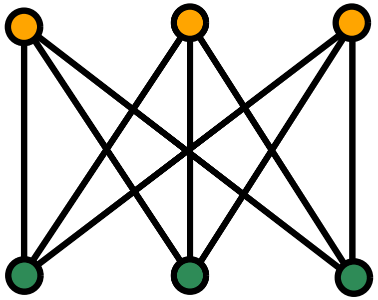

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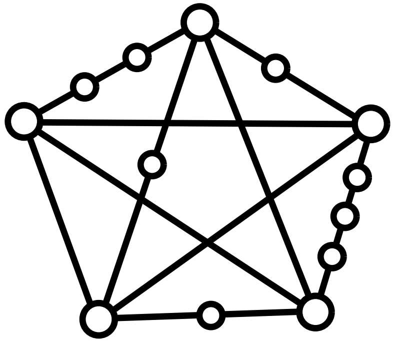

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Kuratowski: any non-planar graph contains a subdivision of K_5 or $K_{3,3}$

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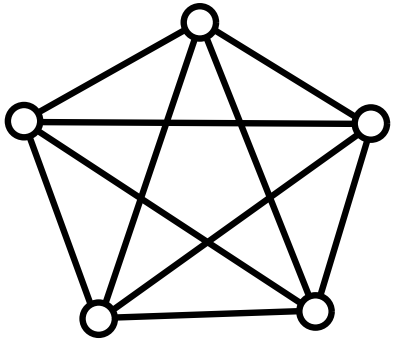

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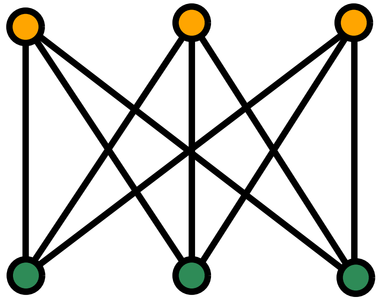

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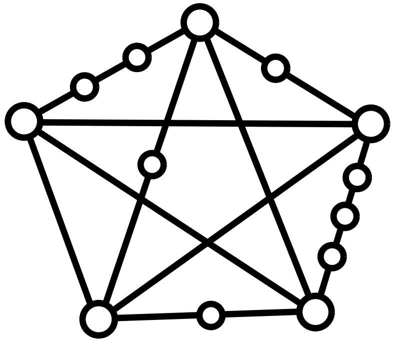

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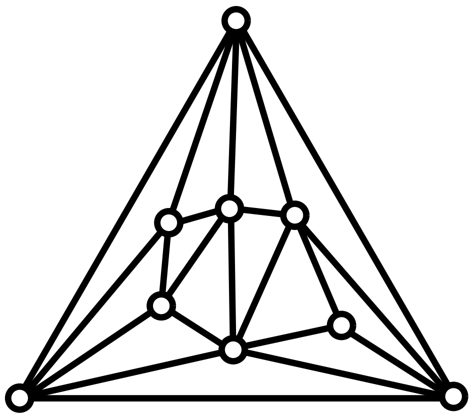

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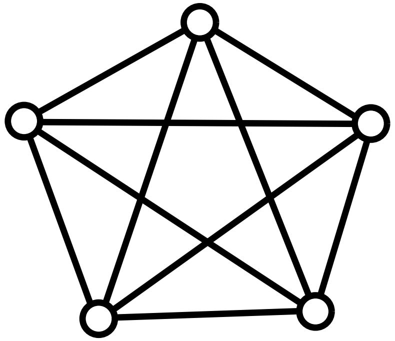
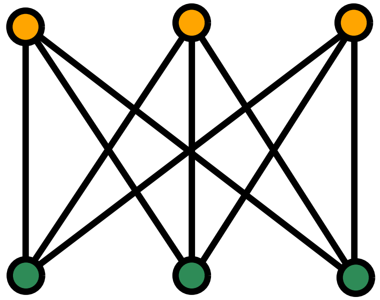
K_5  $K_{3,3}$ 

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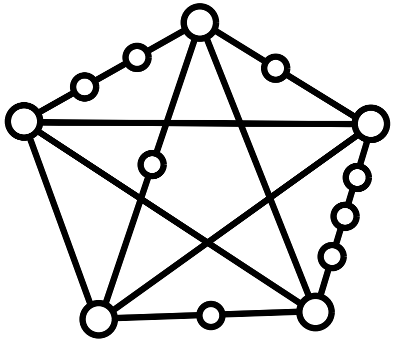

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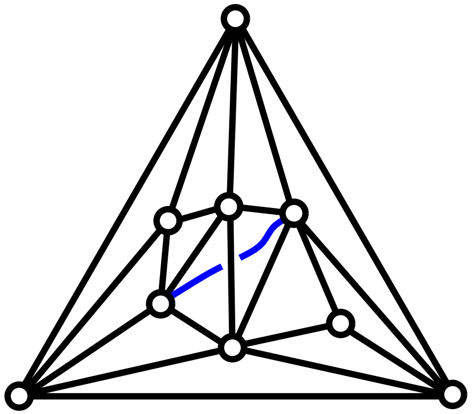

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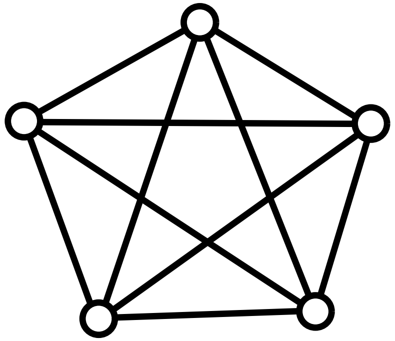

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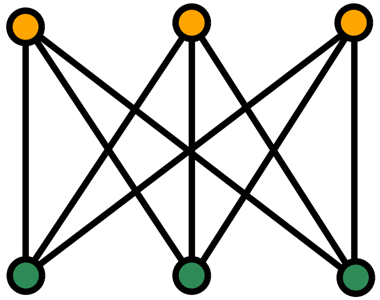

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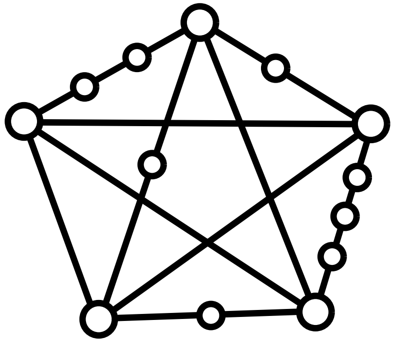

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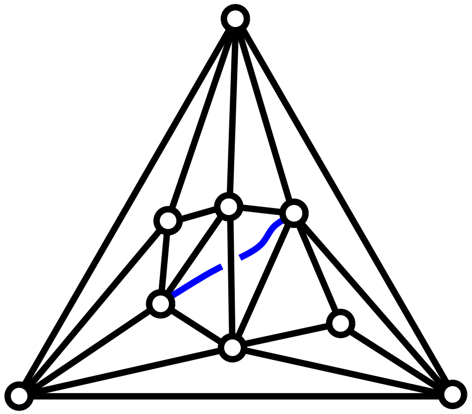

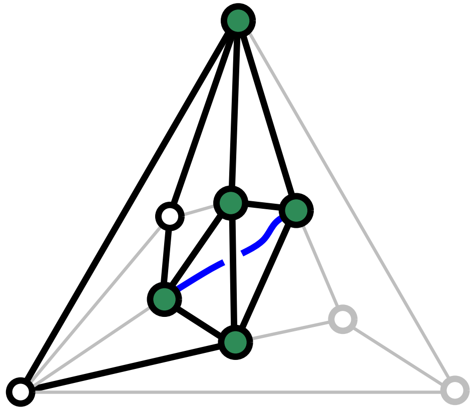

contains subdivision of K_5

k-connectivity in graphs

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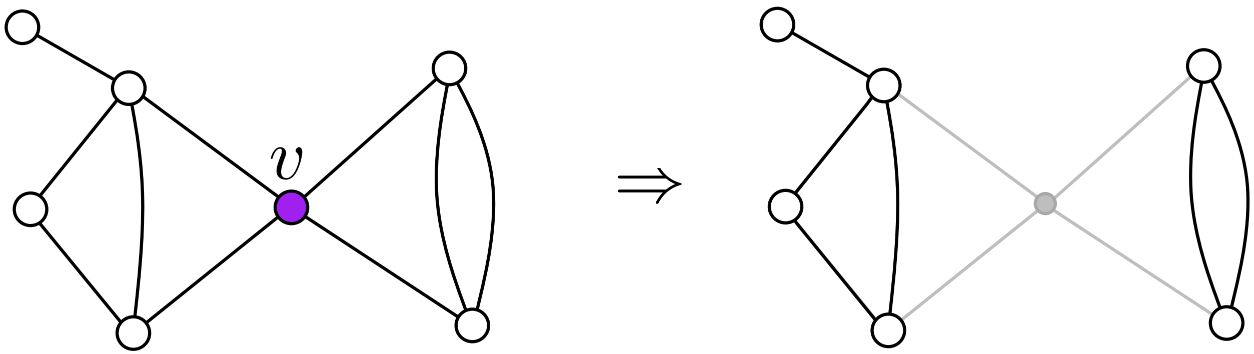
- not 2-connected $\Leftrightarrow$ $\exists$ separating vertex



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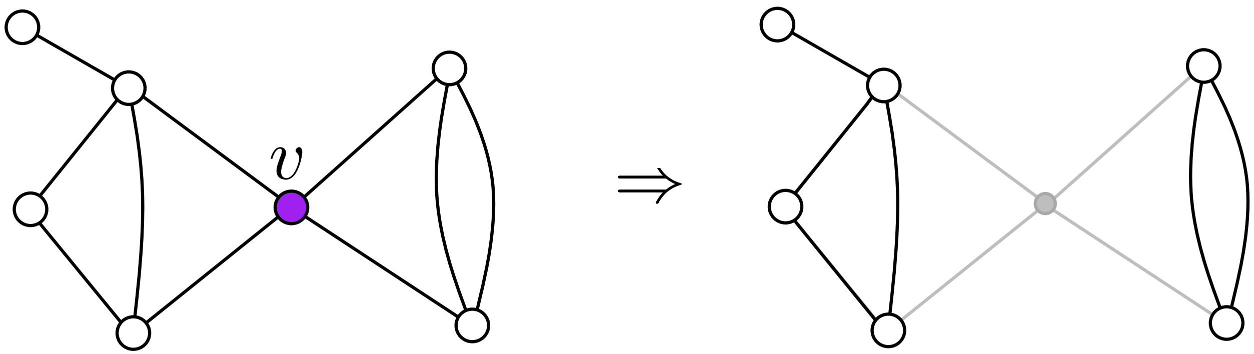
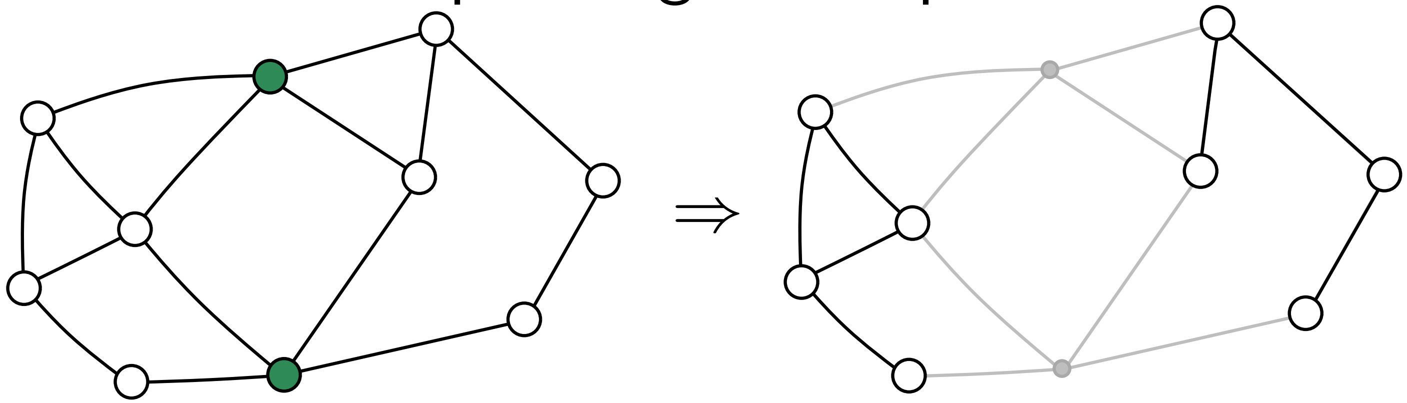
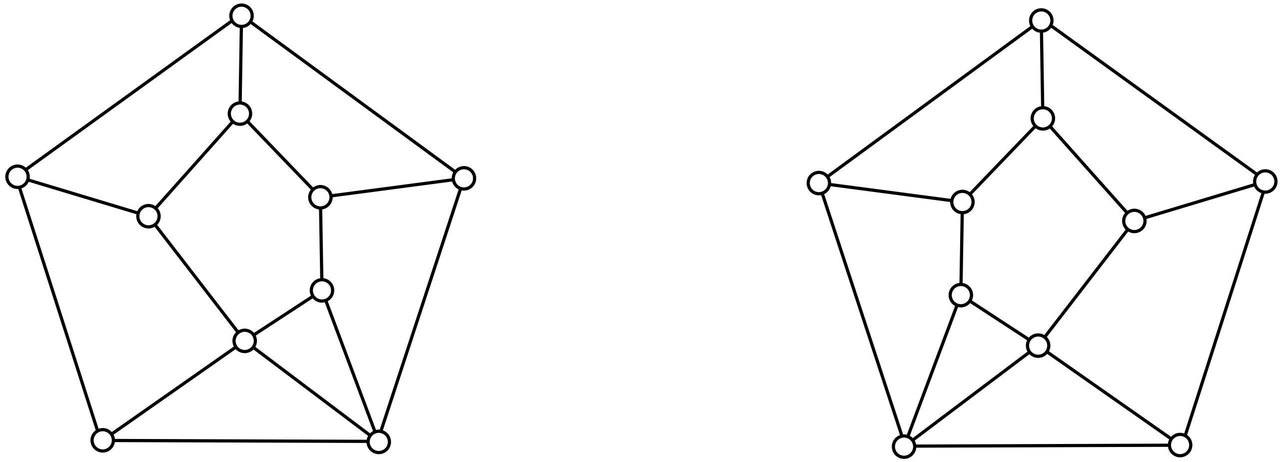


- not 3-connected $\Leftrightarrow \exists$ separating vertex-pair



Whitney's theorem

A 3-connected planar graph has exactly two embeddings on the sphere, which are mirror of each other



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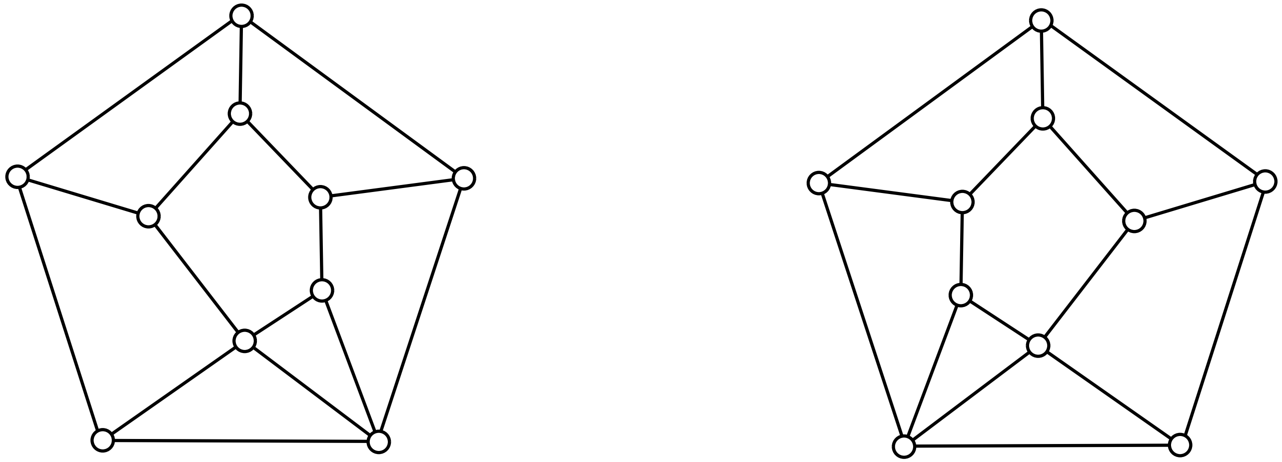
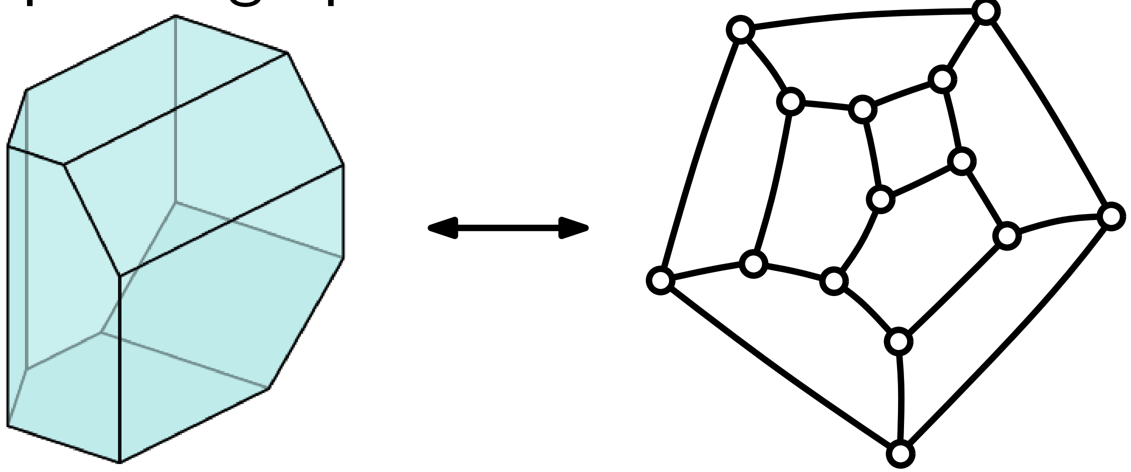


- Other nice feature of 3-connected planar graphs

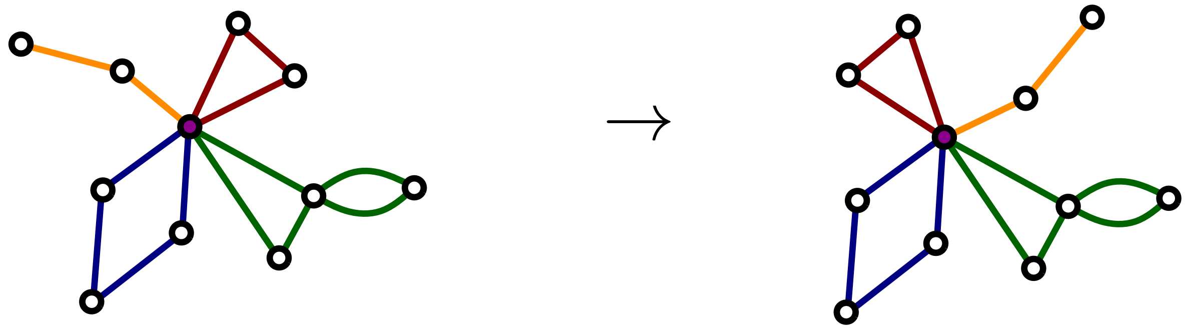
Steinitz'1916: a planar graph is 3-connected iff it can be obtained as the graph of a 3D polytope



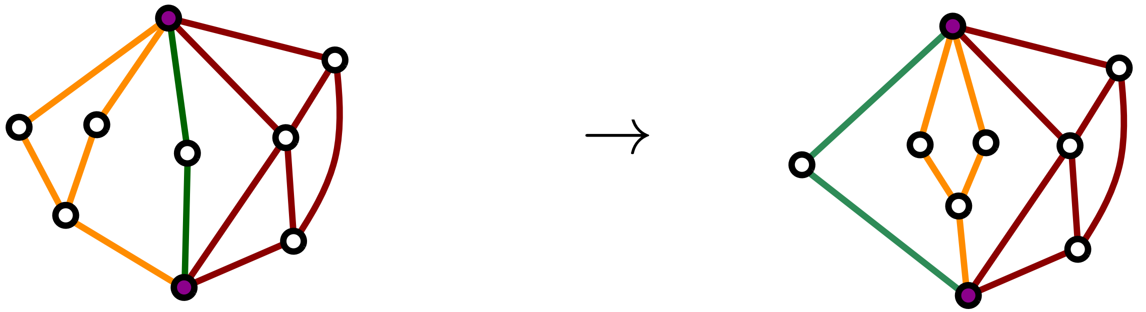
Local operations to change the embedding

Besides taking the mirror image, one can also:

flip at separating vertex (if graph not 2-connected)

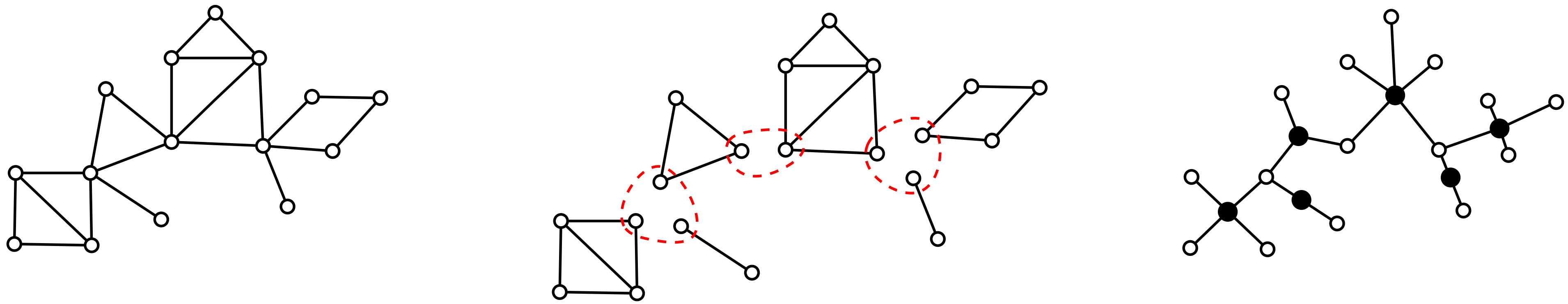


flip at separating pair of vertices (if graph not 3-connected)



Decomposition into 2-c and 3-c components

- Decomposition of connected into 2-connected components [Harary'69]



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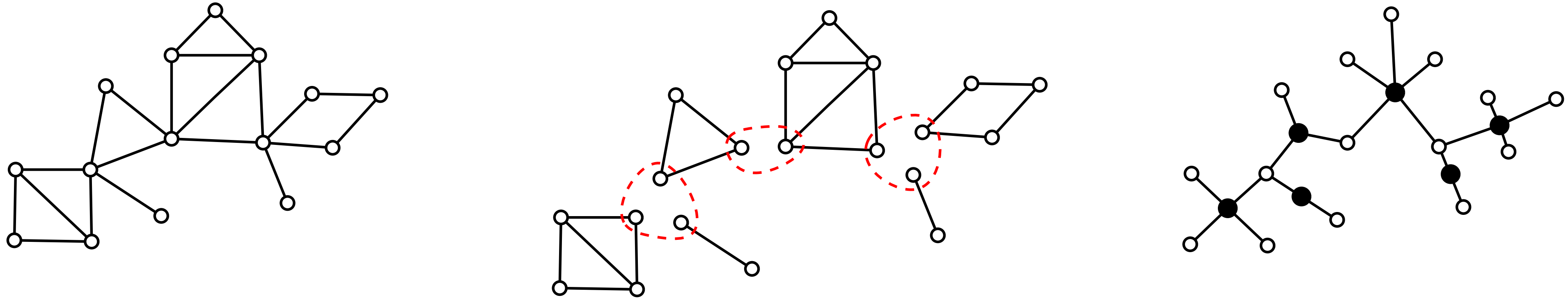
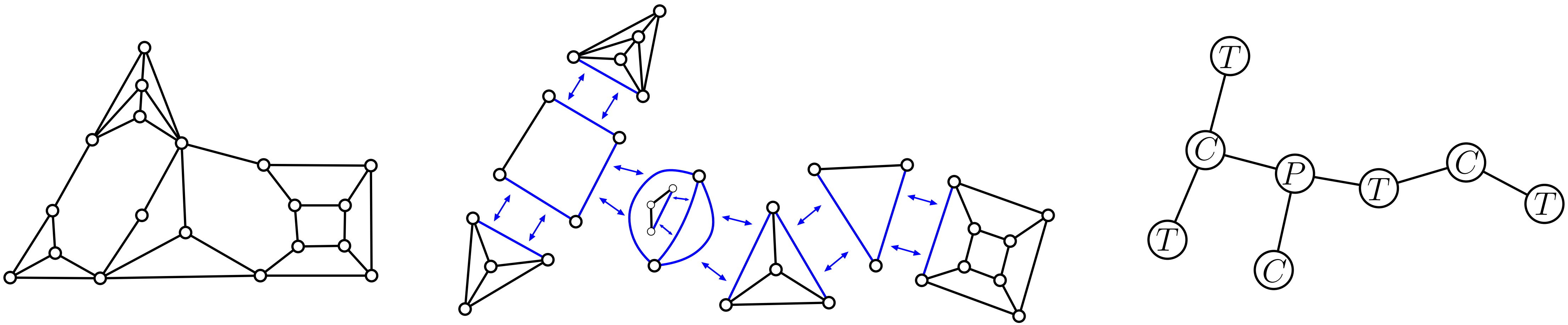


- Decomposition of 2-connected into 3-connected components [Tutte'66]



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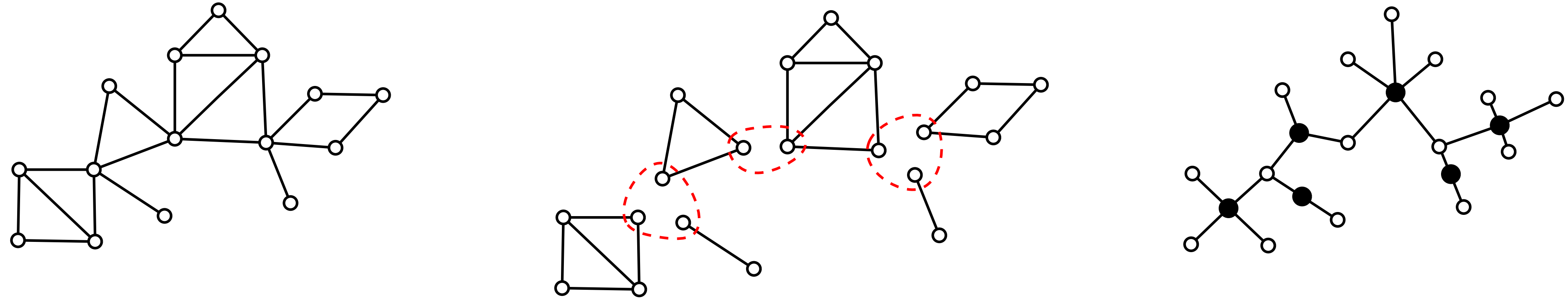
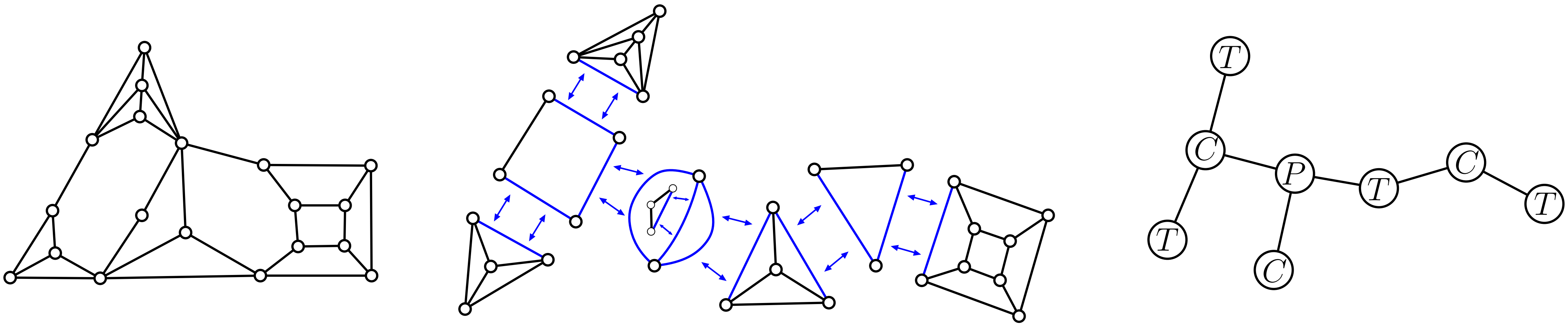


- Decomposition of 2-connected into 3-connected components [Tutte'66]



$\Rightarrow$ captures all the embeddings of a planar graph

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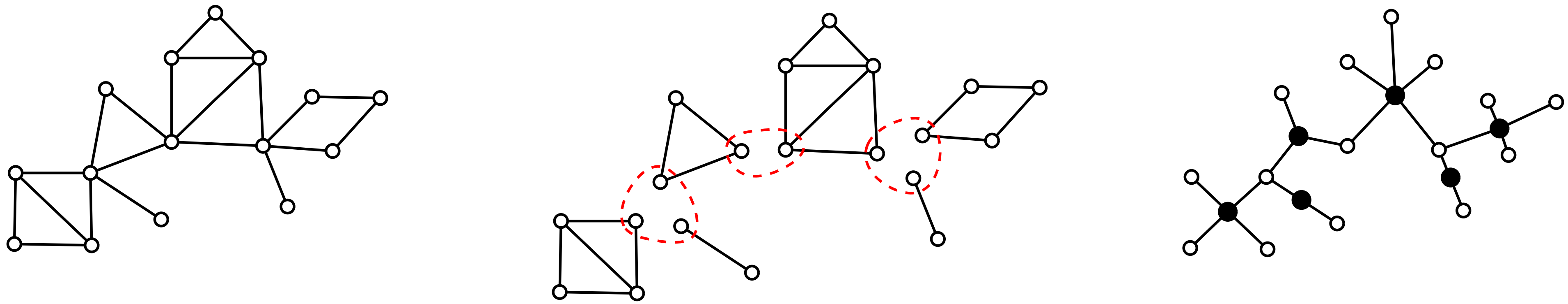

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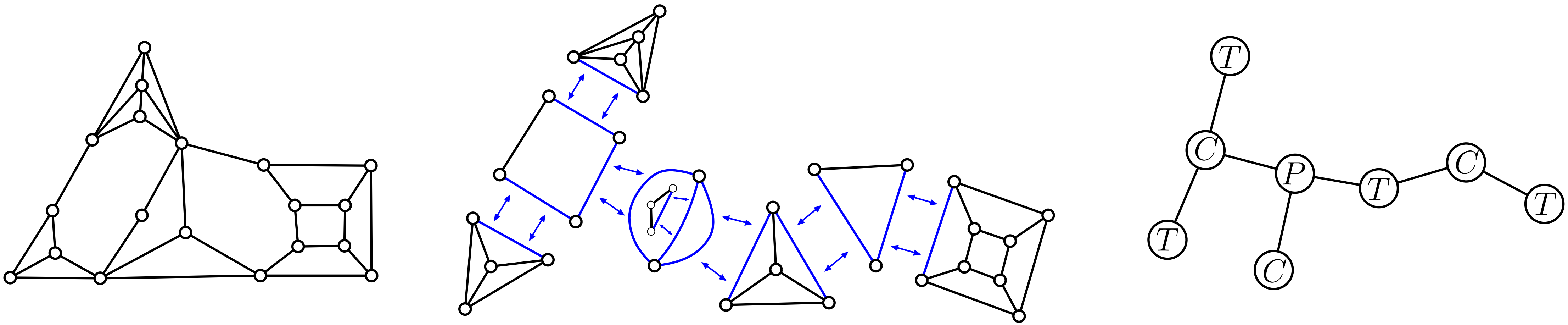


$\Rightarrow$ captures all the embeddings of a planar graph

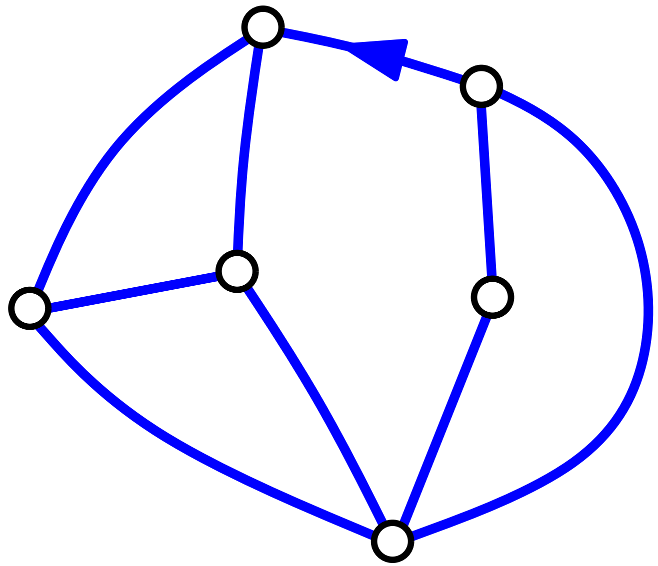
also key tool for the (exact & asymptotic) enumeration of planar graphs,
from enumeration of (3-connected) planar maps [Bender-Gao-Wormald'02, Giménez-Noy'09]

Combinatorial aspects of planar maps

Rooted maps

A map is **rooted** by marking and orienting an edge

a rooted map



the face on the right
of the root is taken
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Rooted maps are combinatorially easier than maps
(no symmetry issue, root gives starting point for recursive decomposition)

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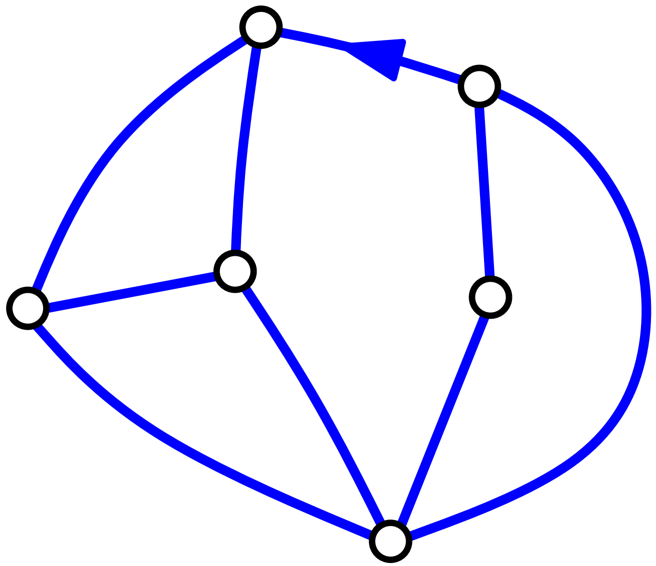

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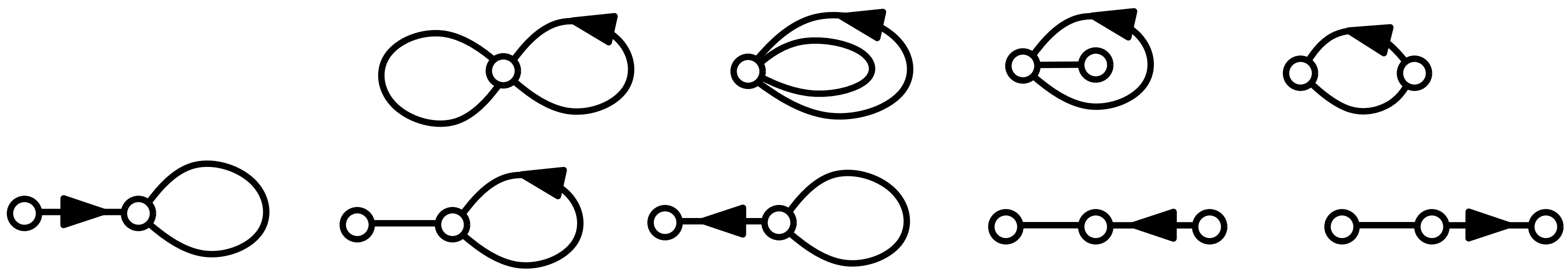
The 2 rooted maps with one edge

The 9 rooted maps
with two edges



Counting rooted maps

Let a_n be the number of rooted maps with n edges

n	1	2	3	4	5	6	7	$\cdots$
a_n	2	9	54	378	2916	24057	208494	$\cdots$

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Theorem: (Tutte'63)

$$\frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$$

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Not an isolated case:

- Triangulations ($2n$ faces)

Loopless: $\frac{2^n}{(n+1)(2n+1)} \binom{3n}{n}$ Simple: $\frac{1}{n(2n-1)} \binom{4n-2}{n-1}$

- Quadrangulations (n faces)

General: $\frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$ Simple: $\frac{2}{n(n+1)} \binom{3n}{n-1}$

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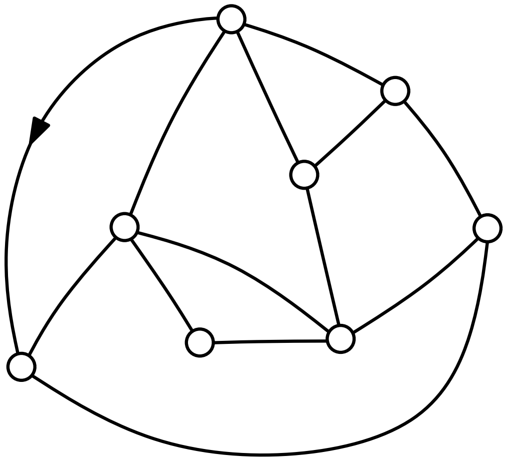
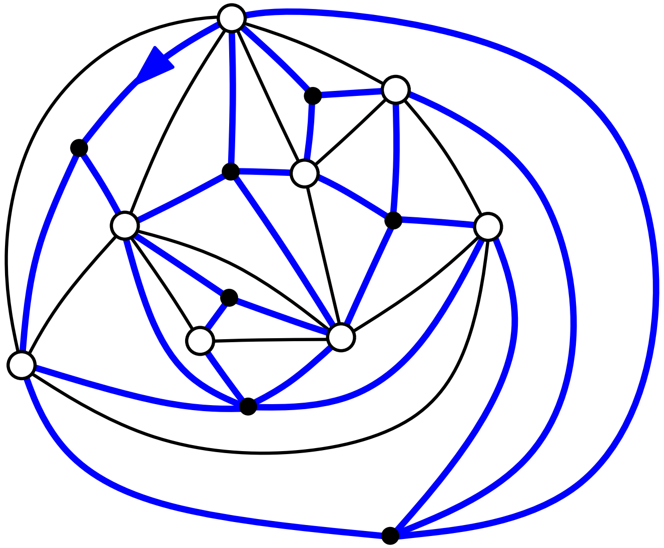
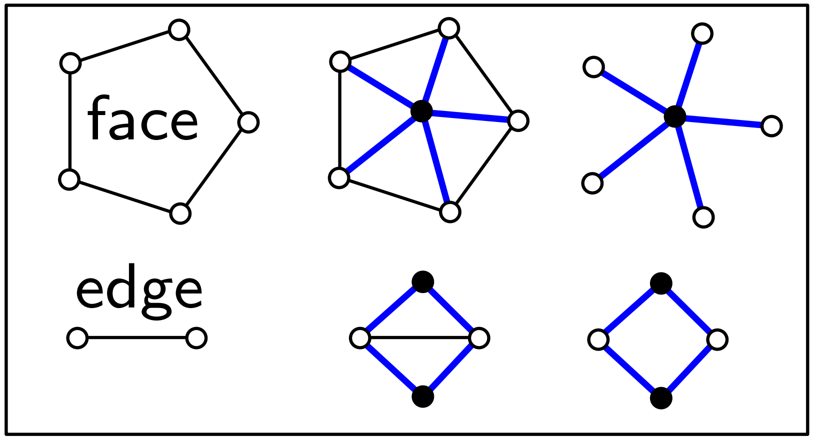
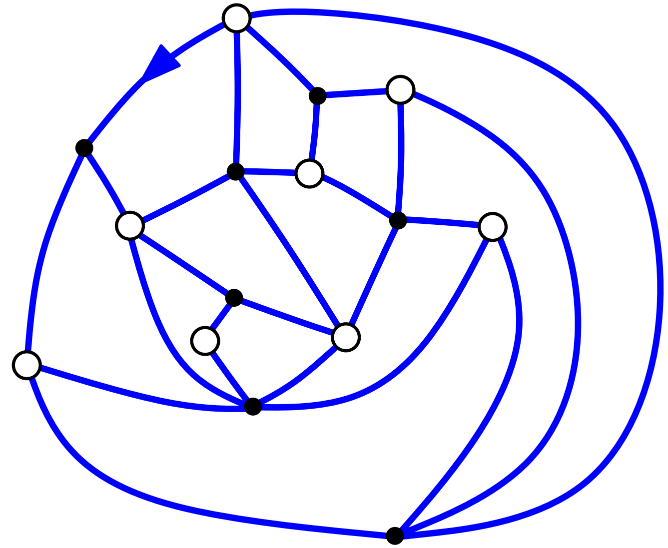
Bijection maps $\leftrightarrow$ quadrangulations



n edges
i vertices
j faces







n faces
i white vertices
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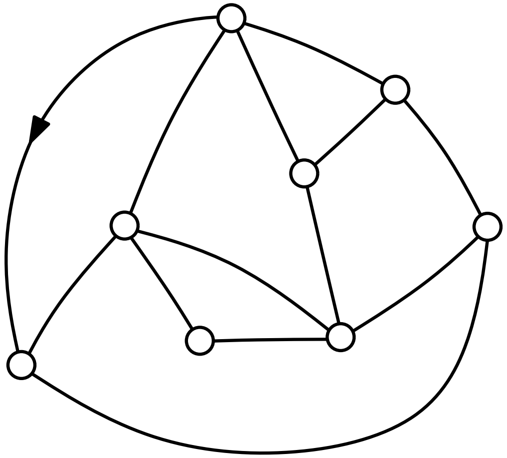

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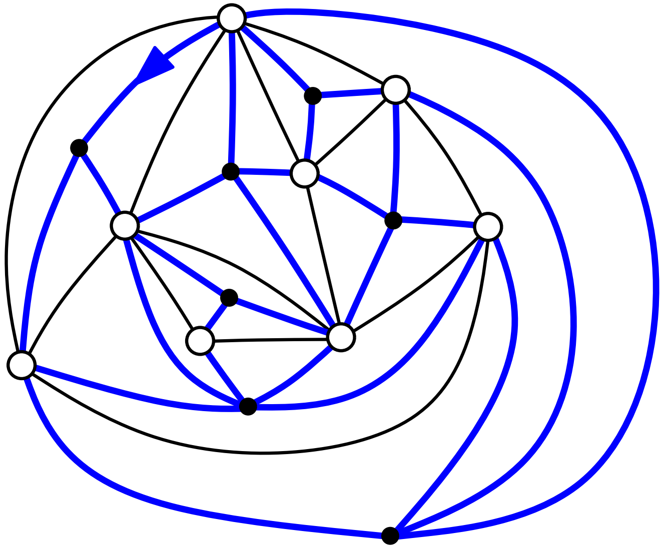

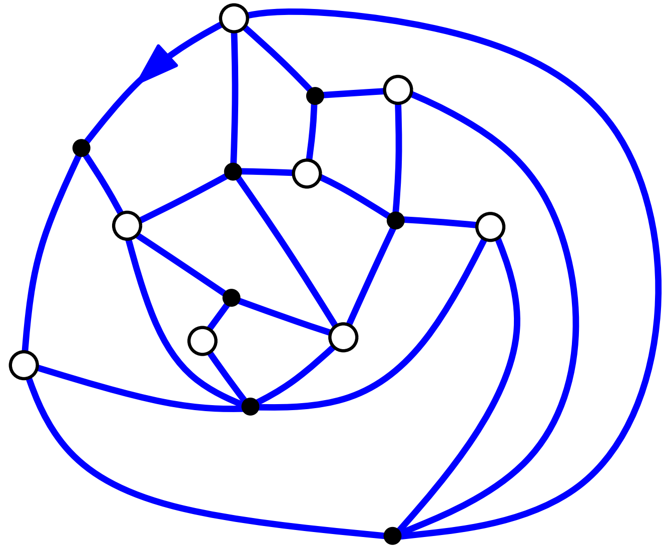

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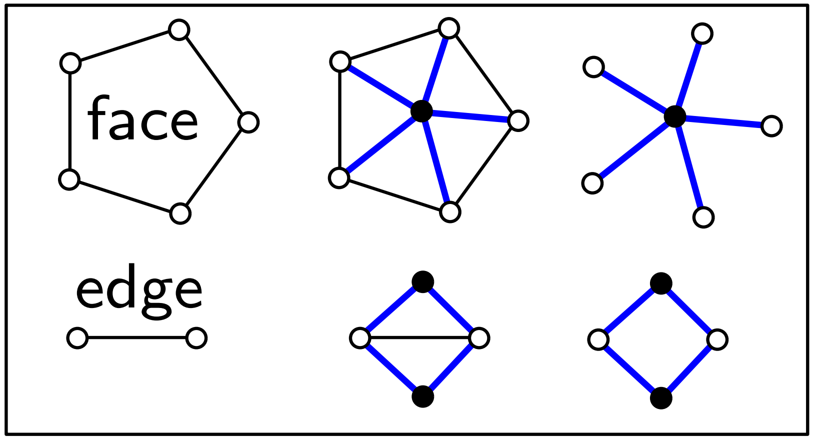
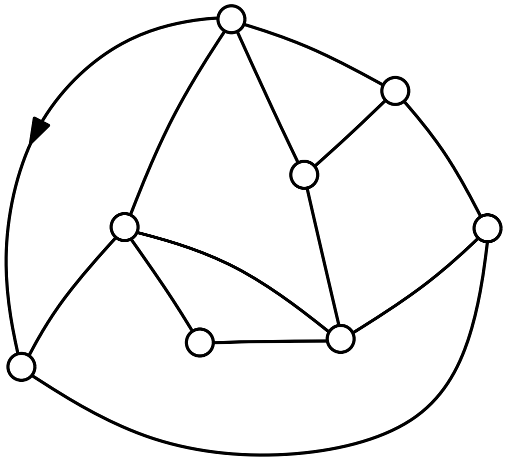


Consequence:

#(rooted maps with n edges) $=$ #(rooted quadrangulations with n faces)

Bijection maps $\leftrightarrow$ quadrangulations



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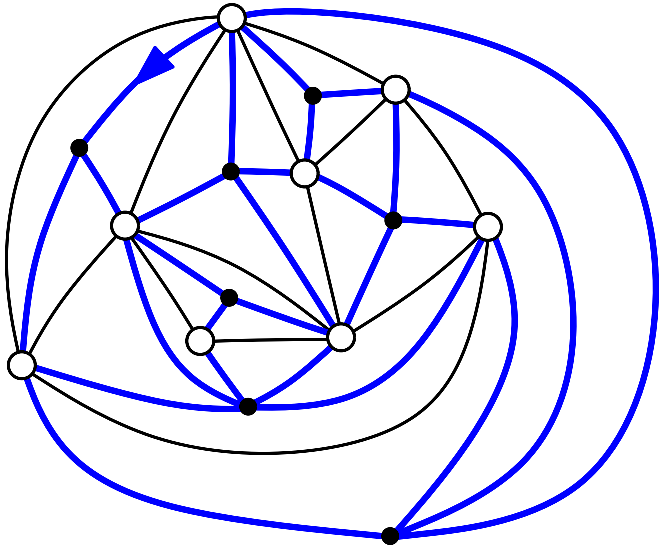

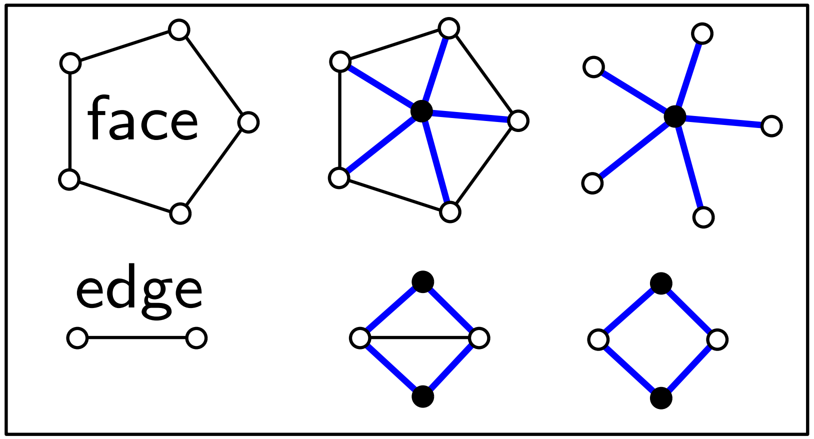

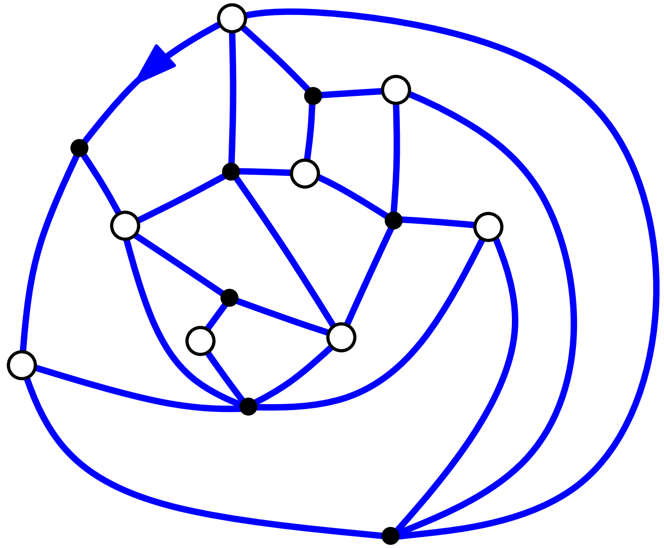

n faces
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Consequence:

#(rooted maps with n edges) $=$ #(rooted quadrangulations with n faces)

It remains to see why this common number is $\dfrac{2 \cdot 3^n}{(n+1)(n+2)}\dbinom{2n}{n}$

Counting methods

- **Generating functions**

  recurrence from root-edge deletion $\Rightarrow$ equations with catalytic variable

  [Tutte'63, Bender&Canfield'86, Bousquet-Mélou&Jehanne'06, Eynard'09]

- **Matrix integrals**

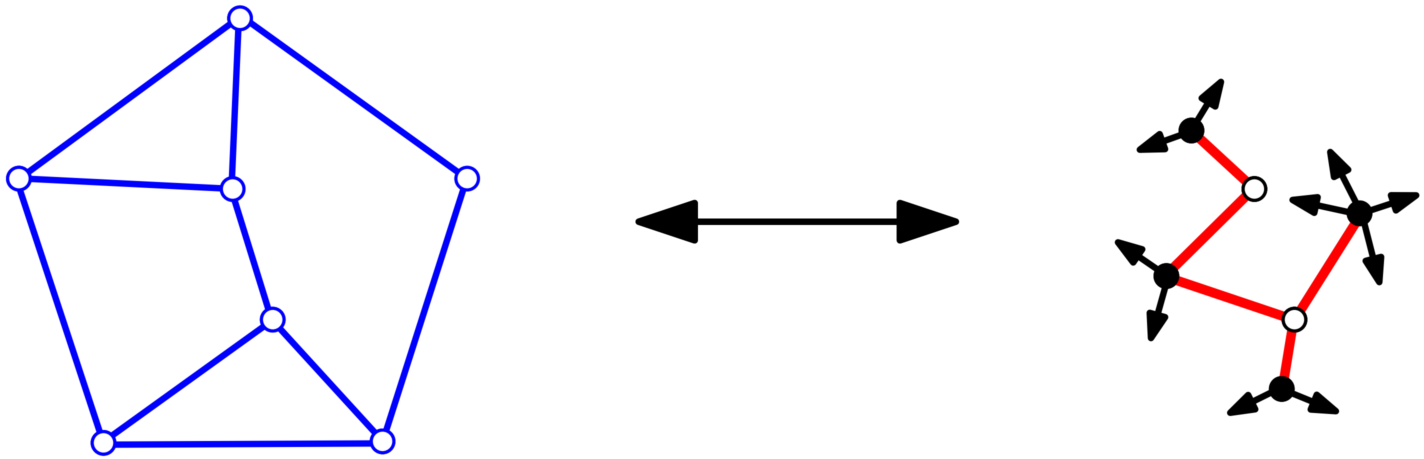
  maps = contributions to certain (gaussian) matrix integrals

  [t'Hooft'74, Brézin et al'78, Di Francesco et al'95]

- **Bijections**

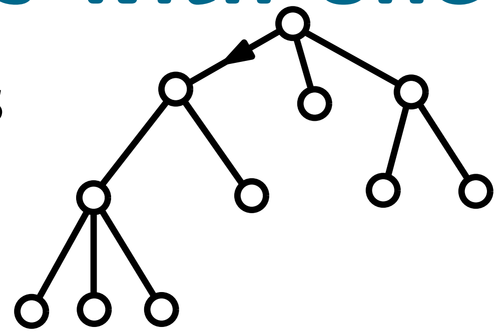
  planar maps $\leftrightarrow$ "decorated" trees

  [Cori-Vauquelin'81, Arquès'86, Schaeffer'97,Poulalhon-Schaeffer'03, Bouttier-Di Francesco-Guitter'04, Albenque-Poulalhon'15]



Counting rooted maps with one face

A rooted map with one face is called a **rooted plane tree**



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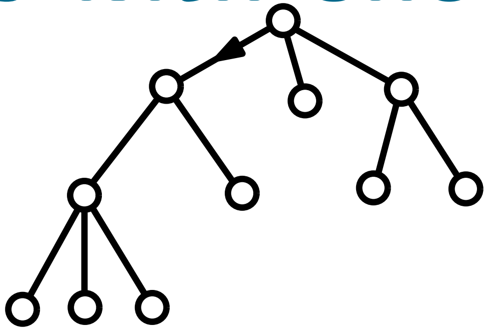

Let c_n be the number of rooted plane trees with n edges

Let $C(z) = \sum_{n \geq 0} c_n z^n$ be the associated generating function

$$C(z) = 1 + z + 2z^2 + 5z^3 + 14z^4 + \cdots$$

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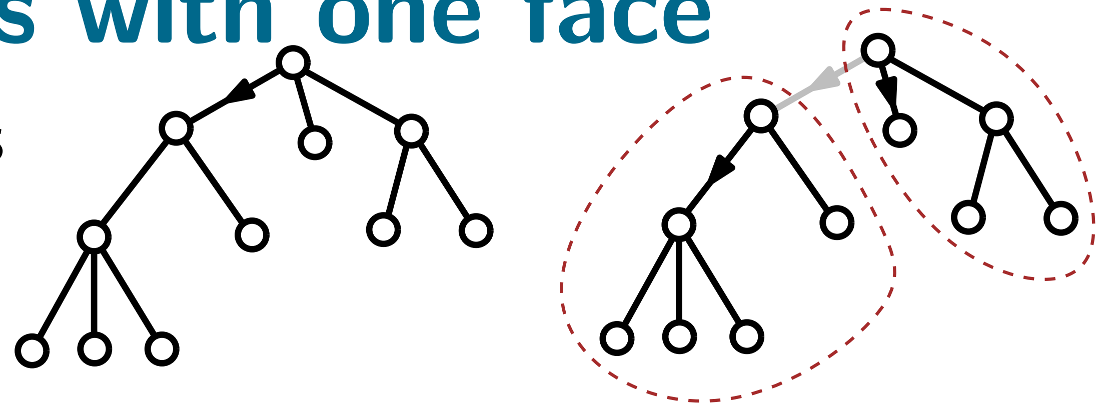

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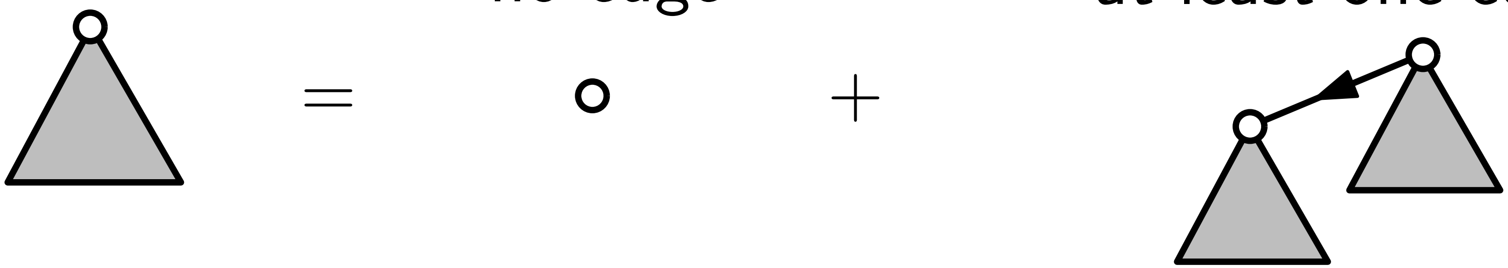
Decomposition at the root:

no edge

at least one edge



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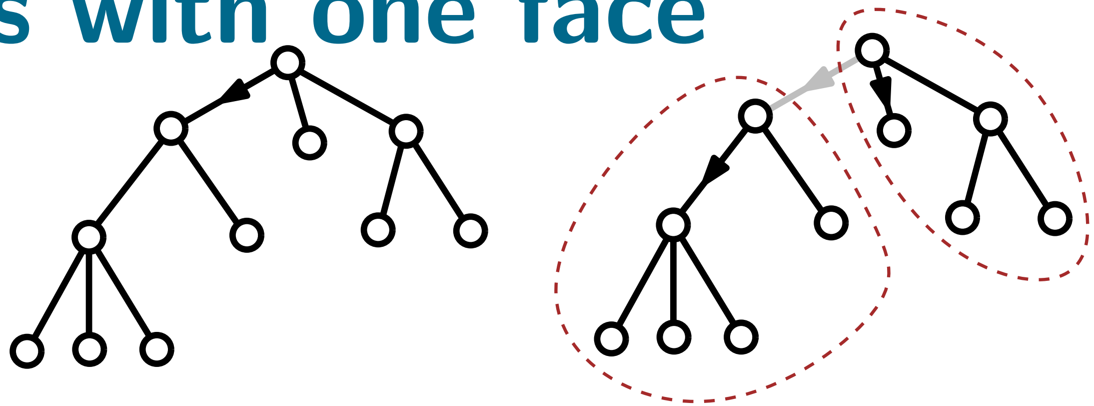

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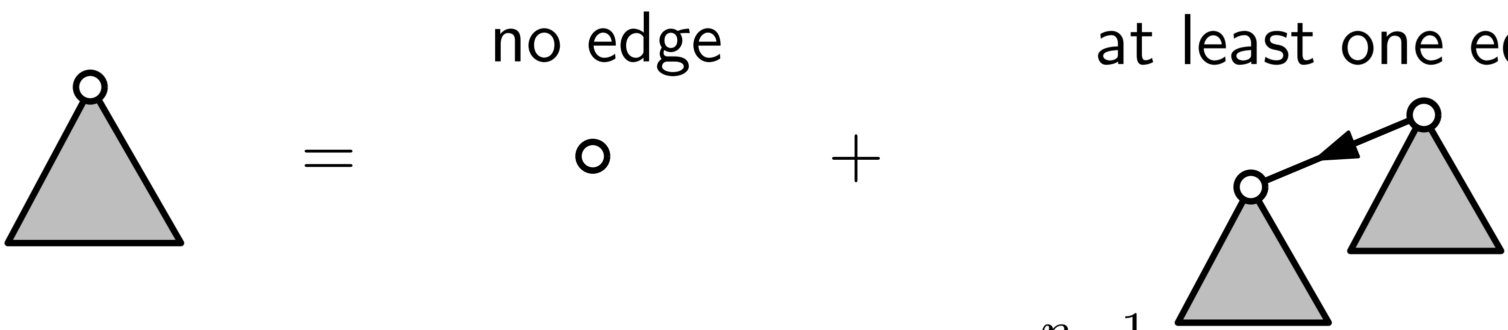

recurrence: $c_0 = 1$ and $c_n = \sum_{k=0}^{n-1} c_k c_{n-1-k}$ for $n \geq 1$

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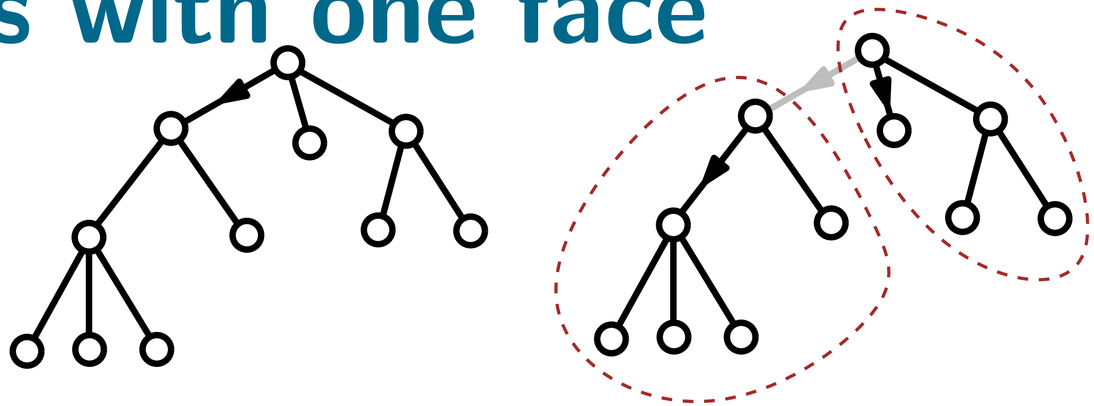

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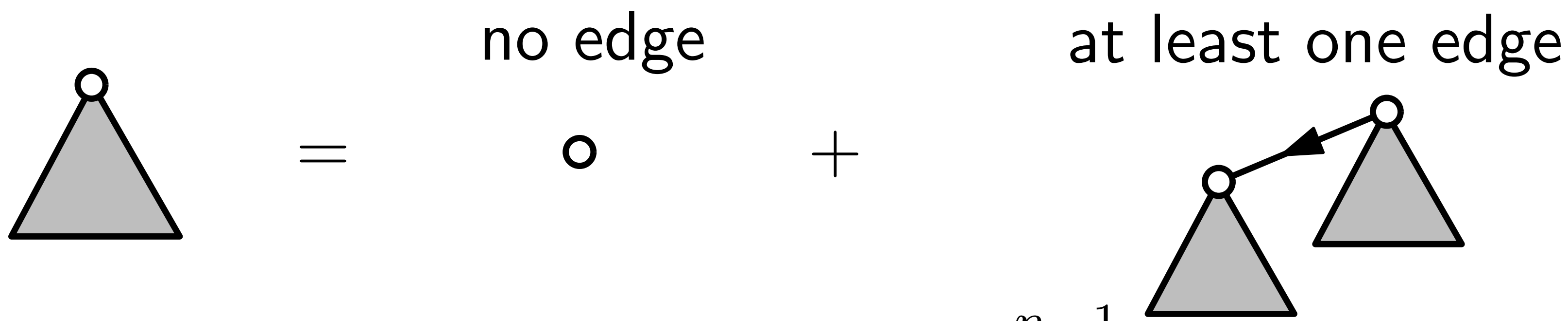

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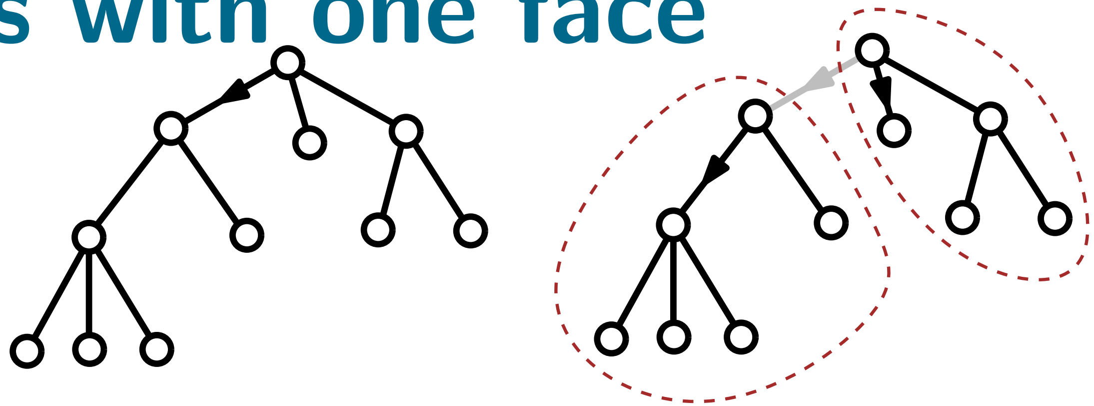

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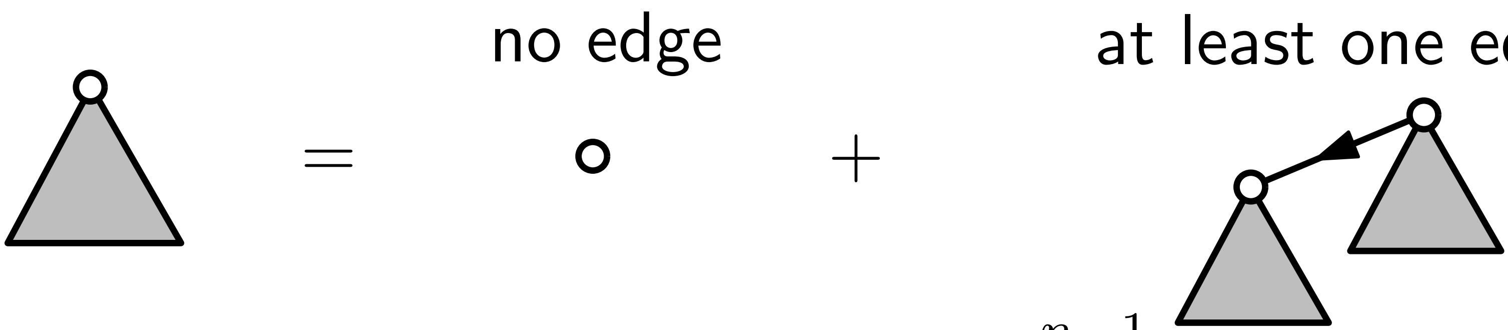

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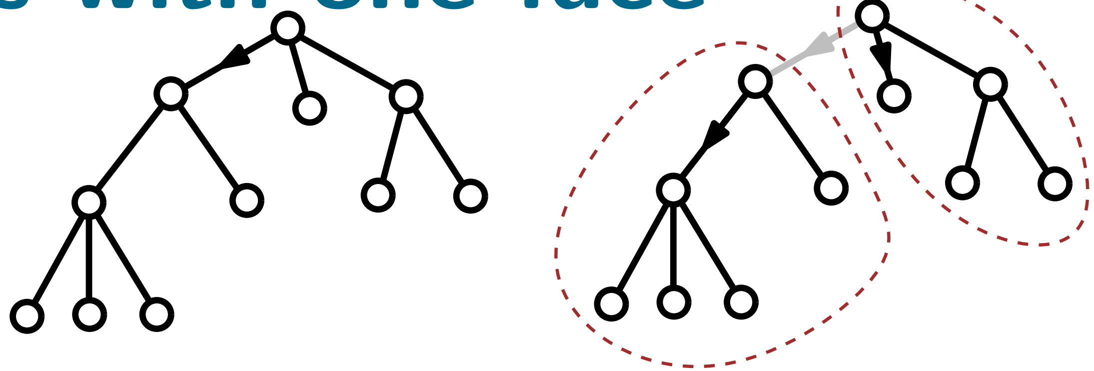

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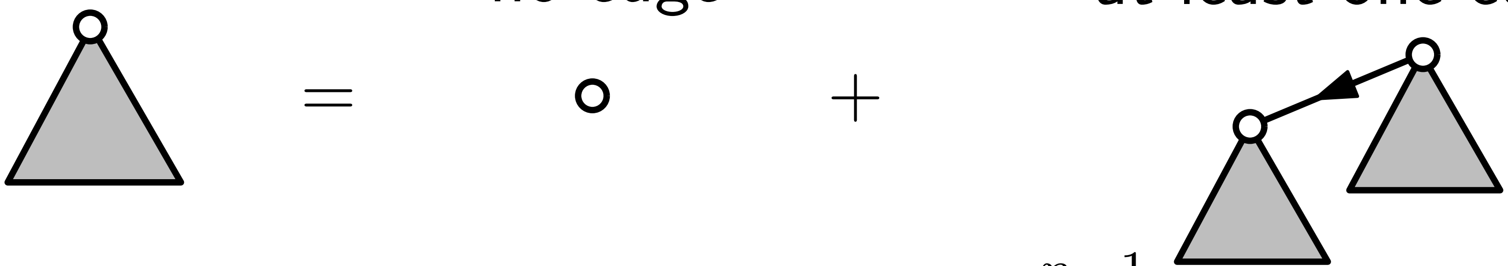
Decomposition at the root:

no edge at least one edge



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Taylor expansion: $C(z) = \sum_{n\geq 0} \frac{(2n)!}{n!(n+1)!}$ $\Rightarrow$ $\boxed{c_n = \frac{(2n)!}{n!(n+1)!}}$ Catalan numbers

Adaptation to rooted maps

Let m_n be the number of rooted maps with n edges

Let $M(z) = \sum_{n\geq 0} m_n z^n$ be the associated generating function

$$= 1 + 2z + 9z^2 + 54z^3 + 378z^4 + 2916z^5 + \cdots$$

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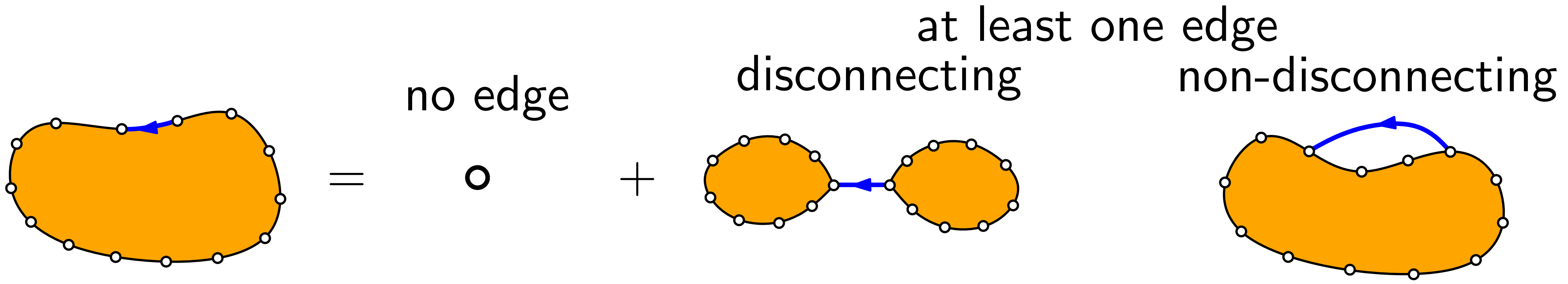

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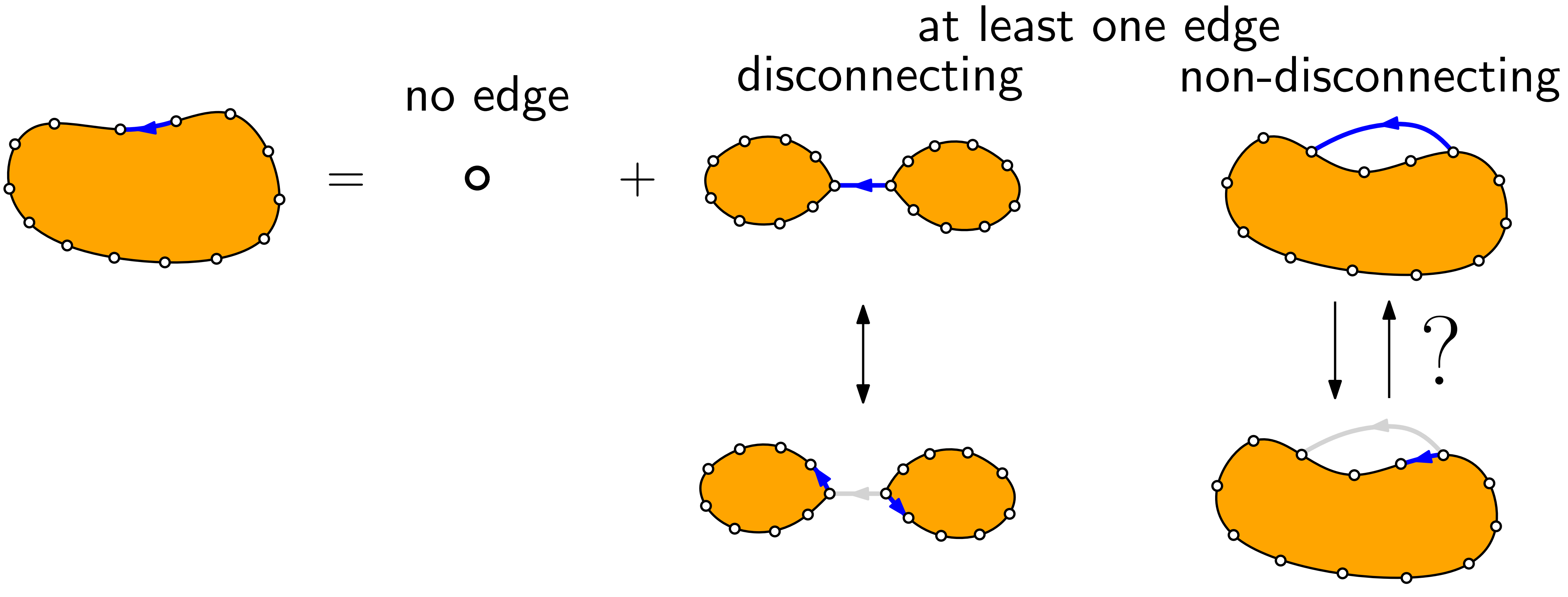

$$M(z) \quad = \quad 1 \quad + \quad zM(z)^2 \quad + \quad ?$$

Adding a secondary variable

Let $m_{n,k}$ be the number of rooted maps with n edges and outer degree k

Let $M(z,u) = \sum_{n,k\geq 0} m_{n,k} z^n u^k$ be the associated generating function

$$= 1 + z(u + u^2) + z^2(2u + 2u^2 + 3u^3 + 2u^4) + \cdots$$

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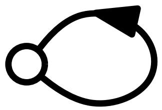
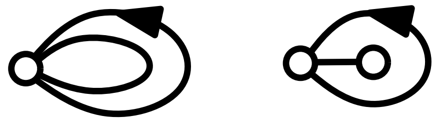
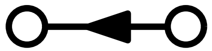
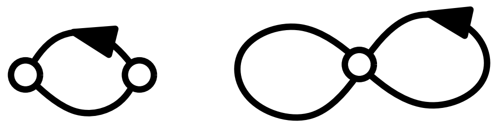
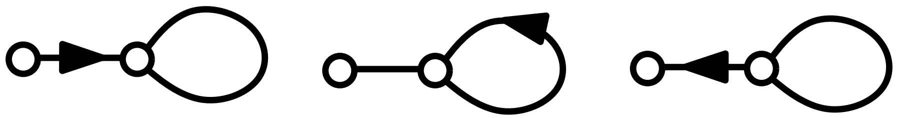
$n = 1$	$n = 2$
$m_{1,1} = 1$ 	$m_{2,1} = 2$ 
$m_{1,2} = 1$ 	$m_{2,2} = 2$ 
	$m_{2,3} = 2$ 
	$m_{2,4} = 2$ 

Adding a secondary variable

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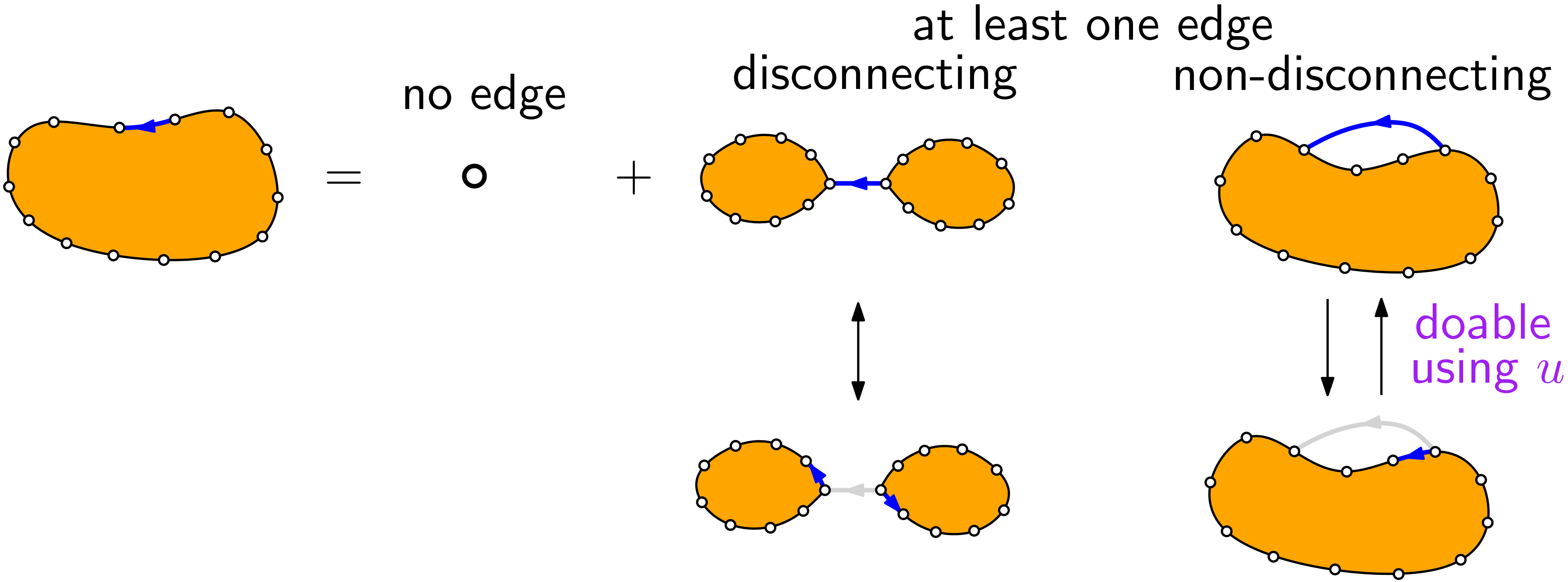

$$M(z,u) \quad = \quad 1 \quad + \quad zu^2 \cdot M(z,u)^2 \quad + \quad A(z,u)$$

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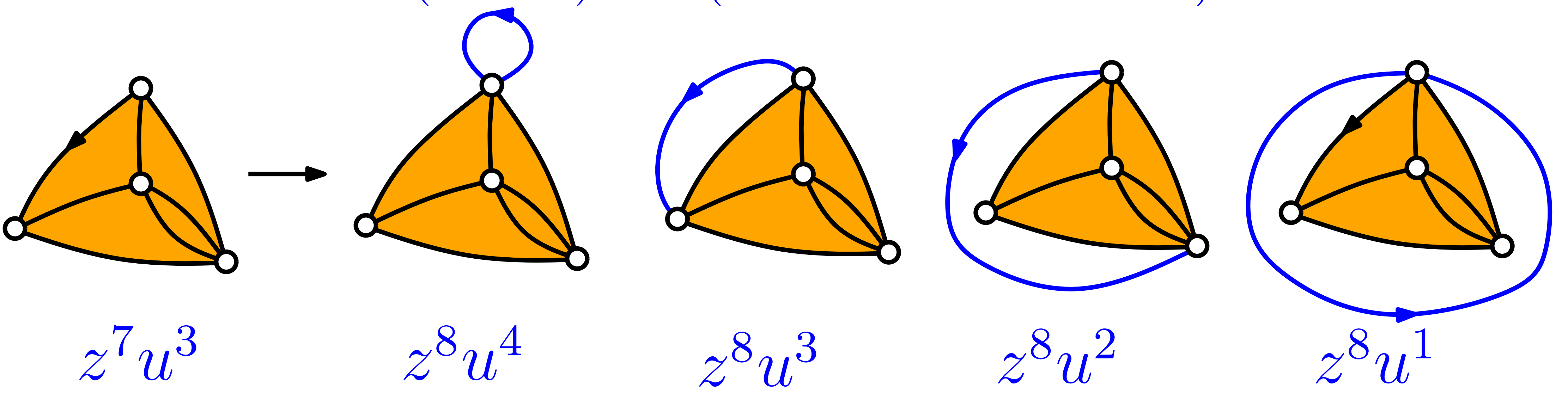

More generally $\quad z^n u^k \to z^{n+1} \cdot (u + u^2 + \cdots + u^{k+1})$

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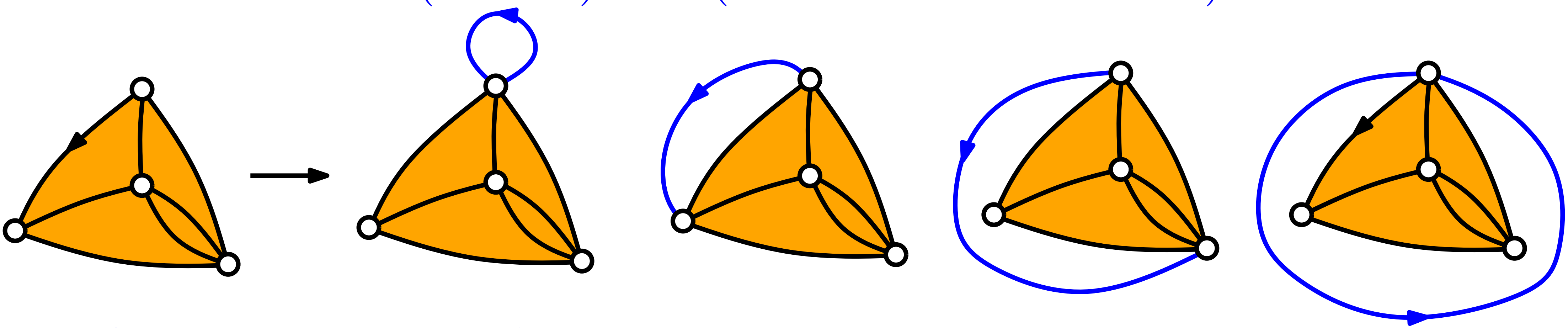

z^7u^3 z^8u^4 z^8u^3 z^8u^2 z^8u^1

More generally $\quad z^n u^k \to z^{n+1}\cdot(u+u^2+\cdots+u^{k+1})$

$$\Rightarrow A(z,u) = \sum_{n,k} m_{n,k}\, z^{n+1}\cdot \underbrace{(u+\cdots+u^{k+1})}_{u\cdot\frac{u^{k+1}-1}{u-1}}$$

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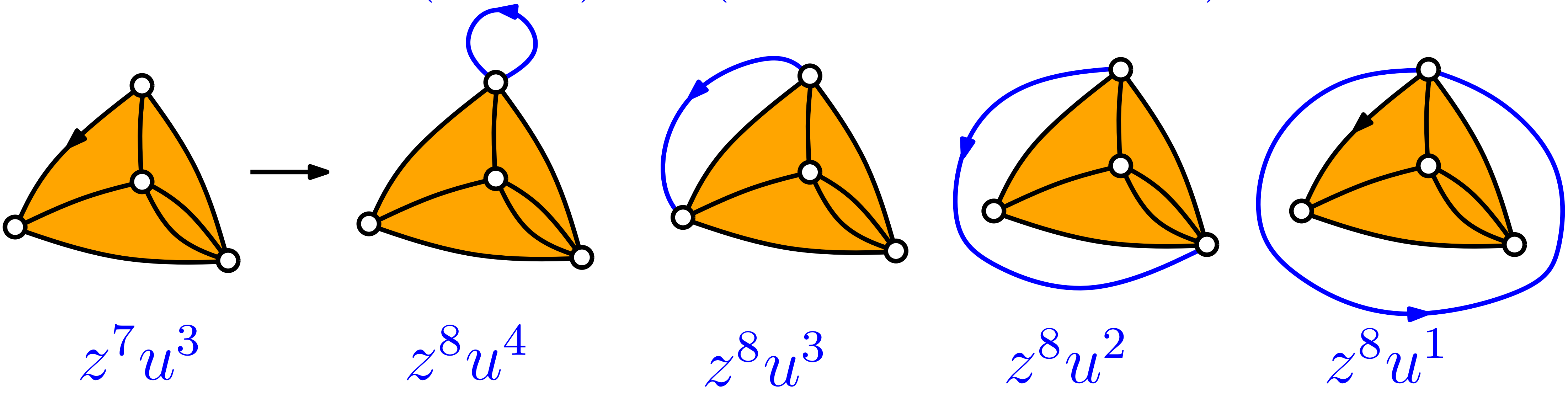

More generally $\quad z^n u^k \to z^{n+1} \cdot (u + u^2 + \cdots + u^{k+1})$

$$\Rightarrow A(z,u) = \sum_{n,k} m_{n,k}\, z^{n+1} \cdot \underbrace{(u + \cdots + u^{k+1})}_{u \cdot \frac{u^{k+1}-1}{u-1}} = zu\frac{uM(z,u) - M(z,1)}{u-1}$$

Adding a secondary variable

Let $m_{n,k}$ be the number of rooted maps with n edges and outer degree k

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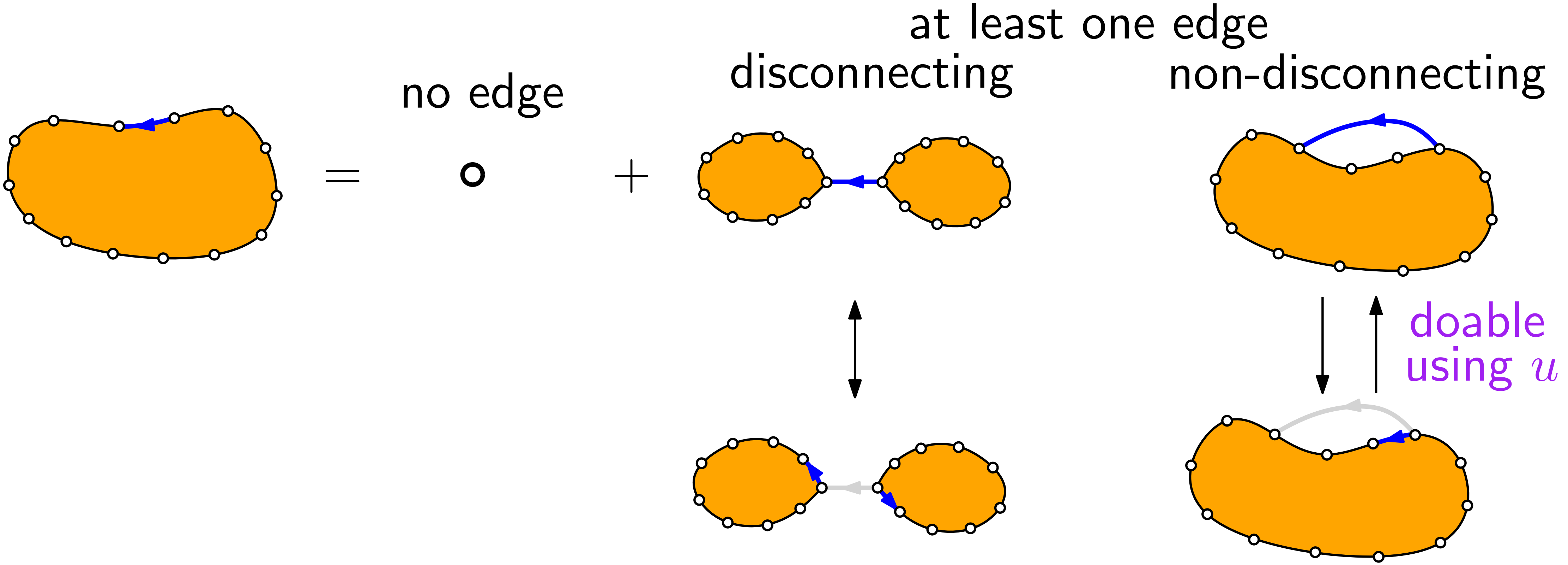
Decomposition by deleting the root:



$$M(z,u) \quad = \quad 1 \quad + \quad zu^2 \cdot M(z,u)^2 \quad + \quad zu\frac{uM(z,u) - M(z,1)}{u-1}$$

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Functional equation obtained:

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of the form $P(M(z,u), M(z,1), z, u) = 0$

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But a unique solution (2 unknown are correlated)

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Guessing/checking or explicit solution methods:

[Brown, Tutte'65, Bousquet-Mélou-Jehanne'06, Eynard'10]

$$\Rightarrow M(z,1) = \frac{1}{54z^2}(-1 + 18z + (1-12z)^{3/2}) = \sum_{n\geq 0} \frac{2\cdot 3^n}{(n+2)(n+1)}\binom{2n}{n} z^n$$