Making a difference

Difference equations as a modelling tool in school mathematics

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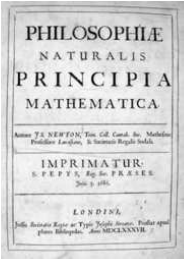
Abstract

The use of difference and differential equations in the modelling is a topic usually studied by advanced students in mathematics. However difference and differential equations appears in the school curriculum in many direct or hidden way. Difference equations first enter in the curriculum when studying arithmetic sequences. Moreover Newtonian mechanics provides many examples for differential equations and numeric solution leads to difference equations which can be treated easily with Computer Algebra Systems or even by Dynamic Geometry Softwares. My hypothesis is that numerical methods supported by technology serves a tool which helps the early introduction of modelling concepts.

Outline

The key concept

Newton's second law describes the relationship between the forces acting on a body to the motion of the body. The force might depend on time, position, velocity. In such cases, Newton's law becomes a system of differential equations.



The problem

How to treat it in school physics with school mathematics?

One possible solution

- Transform to difference equations. (Find numerical solutions.) -> mathematical part
- CAS, DG, spreadsheet applications give a new look to difference equations. -> technology part

The Euler method

In the 18th century Leonhard Euler invented a simple scheme for numerically approximating the solution to $y\prime = f(y,t)$, $y(0) = y_0$.

Euler method

$$y_{k+1} = y_k + f(y_k, t_k)\Delta t$$

Symmetric improvement (implicit for y_{k+1})

$$y_{k+1} = y_k + \left(\tfrac{1}{2}f(y_k, t_k) + \tfrac{1}{2}f(y_{k+1}, t_{k+1})\right)\Delta t$$

Improved Euler method

$$y_{k+1} = y_k + \left(\tfrac{1}{2}f(y_k, t_k) + \tfrac{1}{2}f(y_k + f(y_k, t_k)\Delta t, t_{k+1})\right)\Delta t$$

R. Feynman in his famous lectures at Caltech

Meaning of the dynamical equations

$$\mathbf{r}(t+\Delta t) = \mathbf{r}(t) + \Delta t \mathbf{v}(t)$$
$$\mathbf{v}(t+\Delta t) = \mathbf{v}(t) + \Delta t \mathbf{a}(t)$$

Therefore, if we know both the **r** *and* **v** *at a given time, we know the acceleration, which tells us the new velocity, and we know the new position–this is how the machinery works. The velocity changes a little bit because of the force, and the position changes a little bit because of the velocity.* [The Feynman lectures on Physics, Chapter 9. Newton's laws of dynamics]

This is the Euler method applied for equations of motion.

Discrete models for equations of motions

Assumption: $\mathbf{F} = \mathbf{F}(\mathbf{r})$, $m = 1$.

Euler first order model

$$\mathbf{r}_{k+1} = \mathbf{r}_k + \Delta t \mathbf{v}_k$$
$$\mathbf{v}_{k+1} = \mathbf{v}_k + \Delta t \mathbf{F}_k .$$

The ‘index transformation trick’

$$\mathbf{r}_{k+2} = \mathbf{r}_{k+1} + \Delta t \mathbf{v}_{k+1}$$
$$\mathbf{r}_{k+2} - \mathbf{r}_{k+1} = \mathbf{r}_{k+1} - \mathbf{r}_k + \Delta t \left(\mathbf{v}_{k+1} - \mathbf{v}_k \right)$$
$$\mathbf{r}_{k+2} = 2\mathbf{r}_{k+1} - \mathbf{r}_k + \Delta t^2 \mathbf{F}_k$$

Discrete models for equations of motions

Symmetric model

$$\mathbf{r}_{k+1} = \mathbf{r}_k + \Delta t \frac{1}{2} \left(\mathbf{v}_k + \mathbf{v}_{k+1} \right)$$
$$\mathbf{v}_{k+1} = \mathbf{v}_k + \Delta t \frac{1}{2} \left(\mathbf{F}_k + \mathbf{F}_{k+1} \right).$$

Physically motivated simplification at classroom level:

1. Assume that the force is constant in the interval $[t_k, t_{k+1}]$:
$$\mathbf{r}_{k+1} = \mathbf{r}_k + \Delta t \mathbf{v}_k + \frac{1}{2} \Delta t^2 \mathbf{F}_k.$$
2. Now, the force is known at $\mathbf{r}_{k+1}$
$$\mathbf{v}_{k+1} = \mathbf{v}_k + \Delta t \frac{1}{2} \left(\mathbf{F}_k + \mathbf{F}_{k+1} \right)$$

Discrete models for the equations of motions

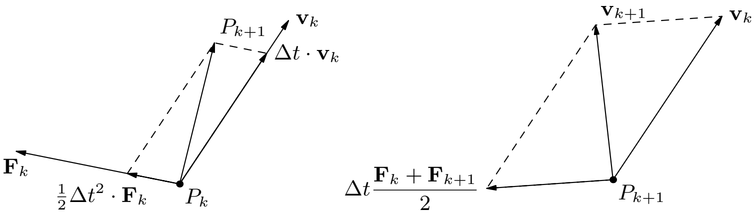


Figure: Construction principle

The 'index transformation trick' gives

$$\mathbf{r}_{k+2} = 2\mathbf{r}_{k+1} - \mathbf{r}_k + \Delta t^2 \mathbf{F}_{k+1}.$$

Note, the improved Euler method gives the same equation.

Comparison of models: The case of the harmonic oscillator

Hook's law

A harmonic oscillator is a system which experiences a restoring force F proportional to the displacement x according to Hooke's law:

$$F = -k \cdot x, \; k > 0$$

In this case we know the exact solution, moreover the symmetric model leads to a system of linear equations which can be easily solved.

Harmonic oscillator

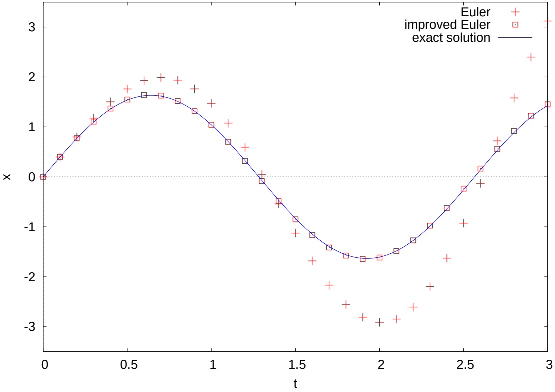


Figure: Comparison of the Euler and the improved Euler model, $D = 6$, $v_0 = 4$, $\Delta t = 0.1$. Plotted by wxMaxima 0.8.2.

Harmonic oscillator

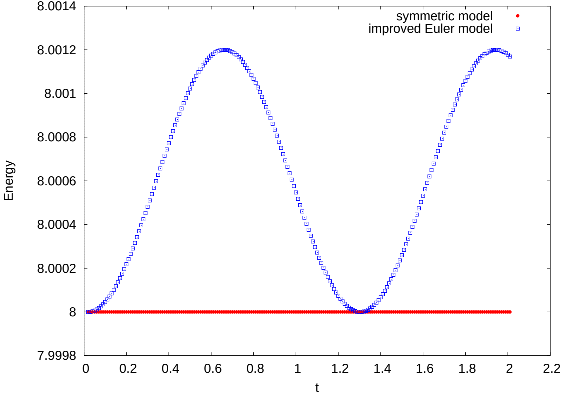
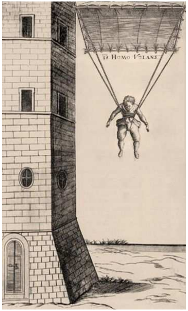


Figure: Comparison of the improved Euler and the symmetric model. $\Delta t = 0.01$. The symmetric model satisfies the energy conservation law.

A CAS example: The terminal speed of parachutist

The terminal velocity of a falling body occurs during free fall when a falling body experiences zero acceleration. This is because of the retarding force known as air resistance. This upward force will eventually balance the falling body's weight.



The terminal speed of parachutist

Exercise

A parachutist of mass m falls freely until his parachute opens. When it is open she/he experiences an upward resistance kv where v is her/his speed and k is a positive constants. Draw the velocity-time diagram and determine the terminal velocity.

The equation of the motion is

$$mv' = mg - kv, \quad v(0) = v_0.$$

We change to discrete model. The Euler first order model:

$$\frac{v_{k+1} - v_k}{\Delta t} = -\frac{k}{m} v_k + g,$$

thus

$$v_{k+1} = \left(1 - \frac{k}{m}\Delta t\right) v_k + g\Delta t.$$

The terminal speed of parachutist

The equation for the symmetric model is given by

$$\frac{v_{k+1} - v_k}{\Delta t} = -\frac{k}{m}\left(\frac{v_{k+1} + v_k}{2}\right) + g.$$

It is then a simple matter to calculate

$$v_{k+1} = v_k \frac{1-\rho}{1+\rho} + \frac{g\Delta t}{1+\rho}, \quad \rho = \frac{k\Delta t}{2m}.$$

The terminal speed of parachutist

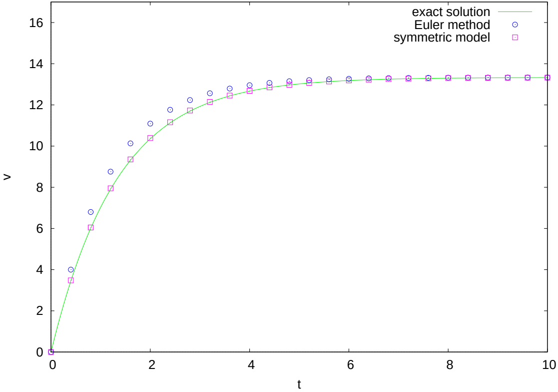


Figure: Motion of parachutist with zero initial speed. Comparison of the Euler method, the symmetric model and the exact solution. $\Delta t = 0.4$. Plotted by Maxima

The terminal speed of parachutist

The standard reasoning for the terminal velocity question: when the parachutist reaches the terminal velocity her/his acceleration is zero, thus

$$v' = 0 \implies v = \frac{mg}{k}.$$

The theoretic value of the terminal velocity tests both models to be acceptable.

Kepler problem (GeoGebra application)

Kepler's laws are concerned with the motion of the planets around the Sun. The Sun (S) is supposed to be in a constant position and the planet (P) moves under the effect of central force:

$$\mathbf{F}(P) = \frac{\kappa}{d(S,P)^3}(S - P).$$

Kepler problem

First method: apply directly the construction principle. GeoGebra's algebraic input is a convenient method to feed difference equations directly.

File View Help

...	Name	Definition	Algebra
1	Point S		S = (0, 0)
2	Point A		A = (10, 2)
3	Vector a		a = (1, 1)
4	Number kappa		kappa = 45
5	Number dt		dt = 0.84
6	Point B	A + dt a + 0.5 kappa / Distance[A, S]³ (S − A) dt²	B = (10.69, 2.81)
7	Vector b	a + dt 0.5 (kappa / Distance[A, S]³ (S − A) + kappa / Distance[B, S]³ (S − B))	b = (0.67, 0.93)
8	Point C	EulerMethod[S, B, b, kappa, dt]	C = (11.13, 3.55)
8	Vector c	EulerMethod[S, B, b, kappa, dt]	c = (0.39, 0.84)

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Figure: GeoGebra's construction protocol for the Kepler problem worksheet. Sixth and seventh lines: P_1 and $\mathbf{v}_1$ from $(P_0, \mathbf{v}_0)$. Tool `EulerMethod` was constructed after the seventh step.

Kepler problem

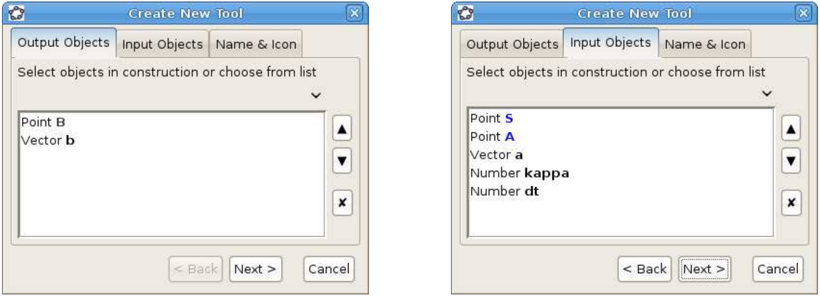

Figure: Creation of a new tool

Kepler problem

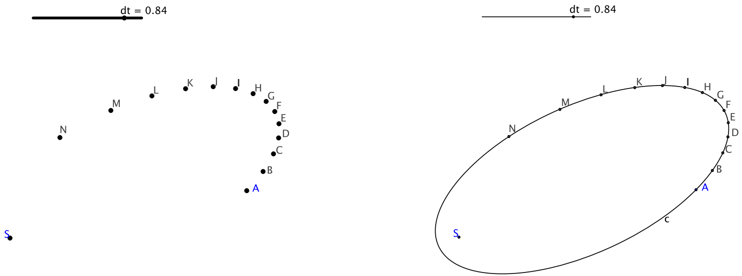

Figure: Created by GeoGebra 3.2.0.0. *S*=Sun, *A*= initial position of the planet. GeoGebra's bonus: the ellipse determined by the first five points.

Kepler problem

Disadvantage of the first method: small $\Delta t \implies$ numerous points.
Second method. Equation

$$\mathbf{r}_{k+2} = 2\mathbf{r}_{k+1} - \mathbf{r}_k + \Delta t^2 \mathbf{F}_{k+1}$$

simplifies the construction pattern. The new spreadsheet feature of GeoGebra supports this reasoning very well.

Kepler problem

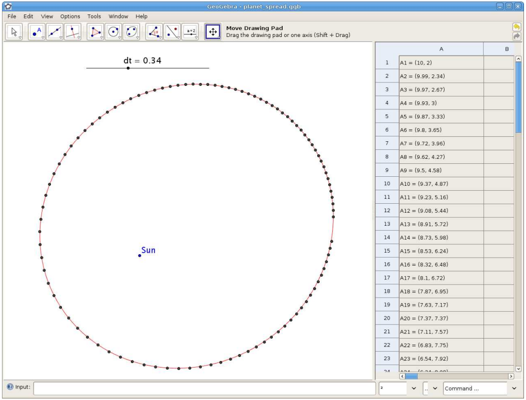

Figure: Kepler problem – created by GeoGebra intensively using the spreadsheet feature. The ellipse was constructed from the first five points with `Conic through Five Points` tool.

Summary

Early introduction of topics related to difference equations should gain more interest in the school mathematics. This idea is not a new discovery, see e.g. L. Berg's papers in the seventies. Nowadays, technology gives a new tool to this concept, not only for the easy numerical calculations but for the graphical representations, too. In my paper I demonstrated how dynamic geometry applications may be used to solve equations of motion numerically. The advantage of the concept is that we can choose complex problems and we can treat them without black boxes. An obvious disadvantage is that the discrete model for the problem is not unique (even it depends on the step-size), so we must test our results.

Summary

Advantages:

1. Newton's second law in action
2. complex 'real life' problem
3. no black-box
4. ...

Disadvantages:

1. Euler method is a first order method
2. The discrete model for the problem is not unique (even it depends on the step-size)
3. ...

Added after the conference

In his keynote lecture at the conference Andre Heck proposed the (two or three dimensional) ‘Mass-on-spring’ oscillator problem for student’s project. The force law is

$$\mathbf{F}(P) = K\frac{d(P,C) - l_0}{d(P,C)}(C - P) + g \cdot (0, -1)$$

where K and g are positive constants, C is a fixed centre, l_0 is the rest length of the spring.

Added after the conference

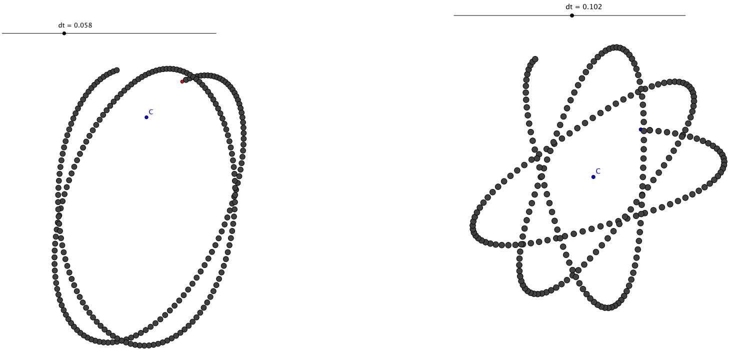

Figure: Mass-on-spring oscillator with non-zero initial velocity. At the right hand side the gravity is zero. Created by GeoGebra using the spreadsheet feature.

Added after the conference

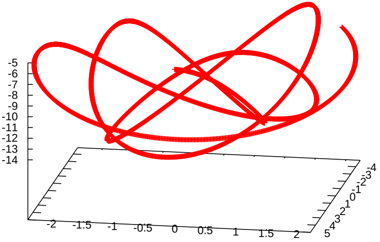

Figure: 3D mass-on-spring oscillator. Plotted by Maxima

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