From Discovery to Proof - a New Approach to an Old Theorem in Plane Geometry

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Table of Content

Table of Content

The Description of the Problem

Take *three* arbitrary points in the Euclidean plane. Using these points we can single out (point to) a fourth point of the plane. What we need to do this?

- First of all a well defined geometric property, which starts with three points, and
- an appropriate geometric construction, which allows an effective drawing of the point.

The point defined by this way is a *center* of the triangle, in the meaning of that geometric property/construction.

The Description of the Problem

Examples:

- I, the incenter, the center of the in-circle, $(X(1))$
- G, the centroid, (center of mass), is the point of concurrence of the medians $(X(2))$
- O, the circumcenter, the center of the circum-circle, or the intersection of the perpendicular bisectors $(X(3))$
- H, the orthocenter, the intersection of the altitudes $(X(4))$
- the center of the nine-point (Feuerbach) circle, $(X(5))$
- the Lemoine point, $(X(6))$
- the Gergonne point, $(X(7))$...

... and also more then 3500 other points! These all are collected in the *Encyclopedia of Triangle Centers* and can be found at the site: http://faculty.evansville.edu/ck6/encyclopedia/ETC.html

The Description of the Problem

Question. (informal level)

Are there a geometric property and a corresponding construction by which we can single out one point, using essentially four points in the Euclidean plane?

Question. (shorter form)

Has the quadrangle any center?

Notice, that an extension of a triangle center – if its defining geometric property works for quadrangles, too – should not be considered (in our approach) as a quadrangle center. Indeed, we seek a property, which is meaningful only for at least four points!

Table of Content

Four-points Theorems

Definition.

Let A, B, C, be three arbitrary points in the plane. The nine-point (or Feurbach) circle of the triangle ABC, is the unique circle defined by the intersection points of the perpendiculars to the sides which passes trough the opposite vertex.

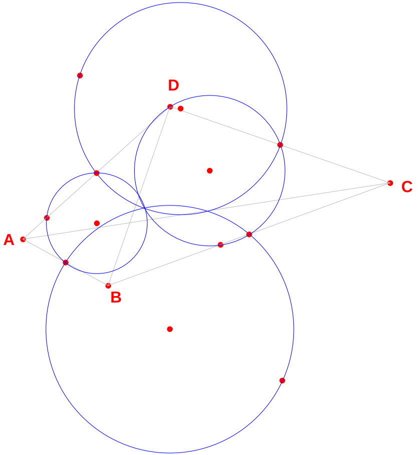
The theorem below is a property valid for four points!

Theorem (A)

Let A, B, C, D be four arbitrary points in the plane. Then the nine-point circles of the ABC, ABD, ACD, BCD triangles have one common point!

The (re-)discovery of this theorem I made by computer experiment: I used the software package Cabri Geometry II.

Four-points Theorems



Let us give point O the name *Feuerbach center.*

Four-points Theorems

The theorem below is also a property valid only for four points!

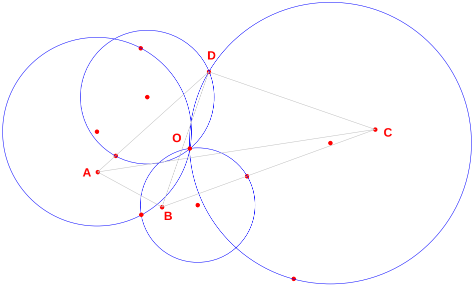
Theorem (B)

Let A, B, C, D be four arbitrary points in the plane. Then the four circles passing on the intersection of the perpendiculars drawn trough the points D, C, B, A to the sides of the triangles ABC, ABD, ACD, BCD respectively have one common point!

This is in some sense a *dual* of the Theorem A.

Of course, I discovered this theorem also by computer, using Cabri Geometry II!

Four-points Theorems



Let us name the point O *dual Feuerbach center.*

Four-points Theorems

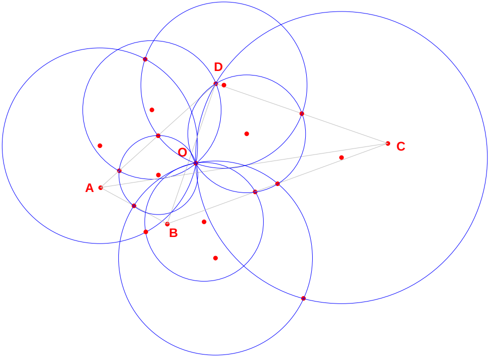
It is quite surprising that the next property is true:

Theorem (C)

The Feuerbach center and the dual Feuerbach center is the same!

This can be checked experimentally using also Cabri Geometry II!

Four-points Theorems



The Feuerbach center and its dual is the same!

Four-points Theorems

Let us consider the particular case, when three out of the four points are collinear.

Theorem (D)

Let A, B, C and D be four points in the plane such that three (A, B, C) are collinear. By this one of the triangles (ABC) defined by these points is degenerate, so they define only three triangles. Then we have:

- *The 3 nine-point circles have one common point O, which is at the same time the intersection of the altitude drawn from D to the line ABC.*
- *The centers A', B', C' of the nine-point circles of the triangles DBC, DAC and DAB and the point O as well are on a circle.*

Four-points Theorems

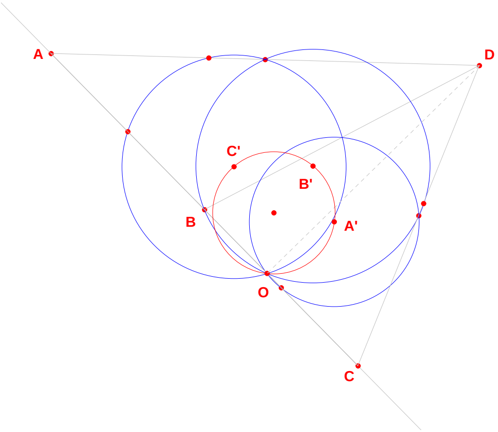


Table of Content

A Computer Algebra

Proofs can be given using Computer Algebra (actually Algebraic Geometry).
Facts used include:

- In the ring of polynomials of several indeterminates every ideal is finitely generated (Hilbert's bases Theorem).
- A Gröbner bases of an ideal is a special bases which preserves most of the properties of modular arithmetic of the integers.
- It is crucial that such a bases can be computed: this is done by the Buchberger algorithm.
- Using coordinates the geometric properties (the assumptions and the consequences) can be expressed by polynomials (polynomial equations) of several indeterminates.

A Computer Algebra


```
// -------------------------------------------
//
// Singular program to prove Theorem A:
//
//
// Let A1, A2, A3, A4 be four arbitrary points
// in the plane.
// Their coordinates are arbitrary variables:
//
// A(u1,u2), B(u3,u4), C(u5,u6), D(u7,u8)
//
// The 9 point circle of triangle BCD is:
// (x-a1)^2+(y-b1)^2-c1^2=0
```

A Computer Algebra


```
ring r=(0,u1,u2,u3,u4,u5,u6,u7,u8),
(a1,a2,a3,a4,b1,b2,b3,b4,c1,c2,c3,c4,x1,x2,y),lp;

poly h1=(u3+u5-2*a1)^2+(u4+u6-2*b1)^2-4*c1*c1;
poly h2=(u3+u7-2*a1)^2+(u4+u8-2*b1)^2-4*c1*c1;
poly h3=(u5+u7-2*a1)^2+(u6+u8-2*b1)^2-4*c1*c1;

poly h4=(u1+u5-2*a2)^2+(u2+u6-2*b2)^2-4*c2*c2;
poly h5=(u1+u7-2*a2)^2+(u2+u8-2*b2)^2-4*c2*c2;
poly h6=(u5+u7-2*a2)^2+(u6+u8-2*b2)^2-4*c2*c2;

poly h7=(u1+u3-2*a3)^2+(u2+u4-2*b3)^2-4*c3*c3;
poly h8=(u1+u7-2*a3)^2+(u2+u8-2*b3)^2-4*c3*c3;
```

A Computer Algebra


```
poly h9=(u3+u7-2*a3)^2+(u4+u8-2*b3)^2-4*c3*c3;

poly h10=(u1+u3-2*a4)^2+(u2+u4-2*b4)^2-4*c4*c4;
poly h11=(u1+u5-2*a4)^2+(u2+u6-2*b4)^2-4*c4*c4;
poly h12=(u3+u5-2*a4)^2+(u4+u6-2*b4)^2-4*c4*c4;

poly h13=(x1-a1)^2+(x2-b1)^2-c1^2-
     ((x1-a2)^2+(x2-b2)^2-c2^2);
poly h14=(x1-a1)^2+(x2-b1)^2-c1^2-((x1-a3)^2+
     (x2-b3)^2-c3^2);

// The conclusion is:
poly g=(x1-a1)^2+(x2-b1)^2-c1*c1;
```

A Computer Algebra


```
// Compute the ideal of hypothesis and conclusion...
ideal I=(1-y*g,h1,h2,h3,h4,h5,h6,h7,h8,h9,h10,h11,
        h12,h13,h14);

// If the Groebner basis is 1, the theorem is proved!
std(I);
//  ----------------------------------------------
```

The computed Gröbner bases is 1, so the theorem is true.

Table of Content

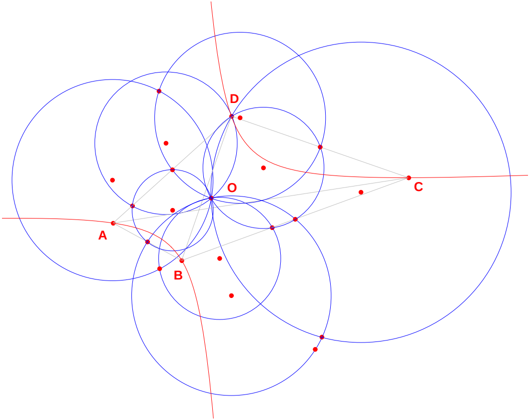
Back to the Roots.

A direct, elementary proof of the Theorem A can be given using the next property:

Theorem (E)

The Feuerbach center of four arbitrary points in the plane is the center of the unique rectangular hyperbola defined by these points!

Back to the Roots.



Back to the Roots.

The elementary proof of the Theorem E is easy. It is based on the next two "Lemma".

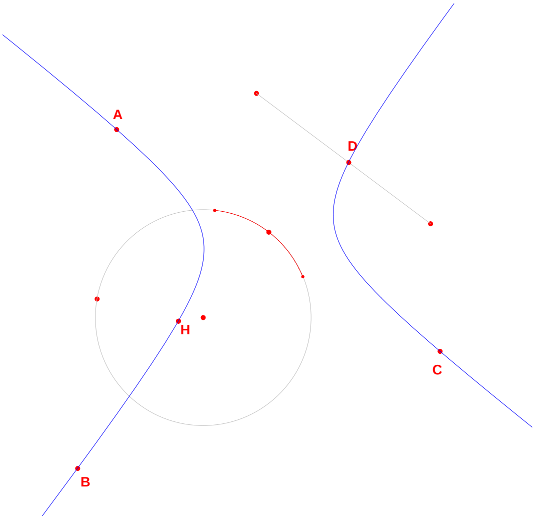
Lemma.

A rectangular hyperbola circumscribed to a triangle passes trough the orthocenter of that triangle.

Lemma. (Brianchon-Poncelet Theorem, 1820)

The center of a rectangular hyperbola circumscribed to a triangle is on the the Feuerbach circle of that triangle.

Theorem of Brianchon-Poncelet.



Back to the Roots.

Proof of the Theorem (A):
Four points in the plane in general position define a unique rectangular hyperbola. As this hyperbola contains all the vertices of the four triangles defined by the given four points, its center lie on all the four nine-point circles. QED.

Remark:
The general position of the points here means that none of the points is the orthocenter of the remaining three.

Conclusions.

Connections to

- the Pappus Theorem,
- the Pascal Theorem,
- the Chasles Theorem,
- the Cayley-Bacharach Theorem,
- ...and the way toward the modern Algebraic Geometry.

Thanks...

Thank You for your attention!